Chapter 9

Time and Variability Indicators, Classical Immunization

9.1. Main time indicators

Knowledge about the indicators of the time structure in the operation

$$O = \{t_h\}\&\{S_h\} \tag{9.1}$$

consisting of receipt (or payment) of amounts $S_1,...,S_n$ to times $t_1,...,t_n$ is important in the management of securities. Thus we preserve the assumption of the same sign into $\{S_h\}$ which are not all zero.[1] Therefore, O results are not fair (see Chapter 4).

Concerning the particular case of a bond, the amounts $\{S_h\}$ are the receipts owed to its owner, both as interest by coupon and as principal by refunds. The payment for the bond purchase is not considered; thus O is a generalized annuity, because the payment schedules can be not periodic.

We will now give a description of *time indicators* useful in financial management. They are in the time dimension, so are measured in the unit chosen in the tickler (usually a year). In addition, they are invariant under proportional

1 As we will see immediately, the time indicators represent "mean times" because they are means of the interval length between the reference instant (in particular, the purchase or evaluation instant) and the maturity of each receipt. Therefore, these time indicators have the feature of "internal means", i.e. are intermediate numbers between the lowest and the highest length of such intervals.

variations of S_h. Then if O is an annuity with constant payments, the indicators for O can be estimated on the corresponding unitary annuity.

9.1.1. *Maturity and time to maturity*

Maturity and time to maturity are the simplest time indicators of O. Using the previous symbols and denoting by t the reference instant (e.g. the purchase or valuation date) the *maturity* of O is t_n, and its *time to maturity* is t_n-t. It is evident that this is an indicator on complete information about the structure of time only on *zero-coupon bonds*, because it neglects the coupon distribution.

With regard to the following indicators, using (7.25), for the sake of simplicity we put at t=0 the reference instant, assuming $t_h \geq 0$, $\forall h$, and at least one $t_h > 0$; thus the time horizon of O is subsequent to 0. Therefore, if t=0 is the purchase or valuation instant of a bond m time units after the issue, this instant is -m and only the payments subsequent to the reference instant are considered. With such an input, the maturity and the time to maturity coincide. It is evident with any t that it is sufficient to use (t_h-t) instead t_h in what follows.

9.1.2. *Arithmetic mean maturity*

This is defined as the arithmetic mean of the maturities t_h, weighted by the amounts S_h of O defined in (9.1), then calculable by the formula

$$\bar{t} = \frac{\sum_{h=1}^{n} t_h S_h}{\sum_{h=1}^{n} S_h} \tag{9.2}$$

The meaning of $\bar{t}$ in terms of mechanics is evident, as the center of mass about the system of S_h put in the points t_h of time axis. Obviously in (9.2) we can assume, instead of S_h, the standardized weights $S_h / \sum_{k=1}^{n} S_k$, that represent the cash-inflow shares at t_h. Then $\bar{t}$ is a synthetic indicator of the cash-flow timing.

9.1.3. *Average maturity*

We define *average maturity* z as the solution of the following equation, referred to (9.1):

$$(1+i)^{-z} \sum_{h=1}^{n} S_h = \sum_{h=1}^{n} S_h (1+i)^{-t_h} \tag{9.3}$$

depending on a given flat-rate *i*. From (9.3) we deduce the explicit form:

$$z = \frac{-\ln\left[\sum_h S_h (1+x)^{-t_h} / \sum_h S_h\right]}{\ln(1+x)} \tag{9.3'}$$

z is an *exponential mean* of t_h, obtained with the transformation of an arithmetic mean by the monotonic function $f(x)=(1+i)^{-x}$. Therefore, it is associative and (9.3) gives $\forall i$ its only solution $z=z(i)$. As $f(x)$ is a discount factor, the average maturity is the time, that, concentrating all payments in this time, we obtain the same present value obtainable according to the given tickler $\{t_h)$.

More generally, if we consider a financial discount law related to the structure of spot prices $v(0,t)$, the average maturity z, depending on $v(0,t)$, is the solution of

$$v(0,z) \sum_{h=1}^{n} S_h = \sum_{h=1}^{n} S_h \, v(0,t_h) \tag{9.3''}$$

The average maturity enables a thorough analysis of the feature and the return of a financial plan made by an operation O^* with amounts of any sign. Sharing the n supplies of O^* according to the amount sign, we obtain the outlays (usually called the *costs* of the plan) and the receipts (also called the *revenues*). Then, for every fixed h,

− if $S_h < 0$, we use $C_h = |S_h| = -S_h > 0$ (cost) and $t_h = t'_r$;

− if $S_h > 0$, we use $R_h = S_h > 0$ (revenue) and $t_h = t''_s$.

Then we obtain the sub-operations $O^{*\prime}$ of the n' costs and $O^{*\prime\prime}$ of the n'' revenues of O^* (being $n'+n'' = n$) in their respective maturities. The value of O^* is the sum of the $O^{*\prime}$ and $O^{*\prime\prime}$ values. Then, using $C = \sum_r C_r$, $R = \sum_s R_s$, and denoting with z_C and z_R the mean maturities of $O^{*\prime}$ and $O^{*\prime\prime}$, and selecting a uniform discount law $v(t)$ (depending only on time t), the O^* value, using the new symbols, is

$$V_0 = -\sum_{r=1}^{n'} C_r \, v(t'_r) + \sum_{s=1}^{n''} R_s \, v(t''_s) = -C\, v(z_C) + R\, v(z_R)$$

Therefore, with the purpose of the valuation, the O^* plan is equivalent to the point input, point output (PIPO) plan $\{z_C, z_R\}\&\{-C,R\}$ obtained by concentrating all costs in z_C and all revenues in z_R . Using $\tau = z_R - z_C$, if $z_C < z_R$ so $\tau>0$, the plan O^* has the *investment feature*, since the costs on average occur before the revenues; but if $z_C > z_R$ i.e. $\tau<0$, instead, the costs on average occur after the revenues, then the plan O^* has the *loan feature.*

In the case of $z_C < z_R$ if we select $v(t)$ subject to strong decomposability, which implies symmetry, then the accumulation factor from z_C to z_R is $v(z_C)/v(z_R)$.

However, in this case, as known, the exchange law is exponential: $v(t) = (1+i)^{-t}$, where i is the interest rate. Then we obtain:

$$V_0 = -C(1+i)^{-z_C} + R(1+i)^{-z_R}$$

where z_C , z_R and τ depend on i. If $i=i^*=$ IRR of O^*, we obtain:

$$V_0(i^*) = (1+i^*)^{-z_R}[-C(1+i^*)^{\tau} + R] = 0, \text{ so: } C(1+i^*)^{\tau} = R$$

This formula clarifies, with reference to the PIPO plan equivalent to O^*, the meaning of the internal rate of return IRR and of the average time length τ.

9.1.4. *Mean financial time length or "duration"*

Given a term structure, defined by spot prices $v(0,t_h)$ in the valuation time 0 and an operation O set as (9.1), we define *duration*, denoted by D (see Macaulay, 1938) in a reference time put in 0, the *arithmetic mean of times t_h weighted by the present values $S_h \cdot v(0,t_h)$* of amounts S_h, that is *by the prices at* 0 of the *zero coupon bonds (*ZCB) that enable the buyer of the bonds to receive S_h at the times t_h, (h=1,...,n). Then the duration is univocally obtained by

$$D = \frac{\sum_{h=1}^{n} t_h S_h v(0,t_h)}{\sum_{h=1}^{n} S_h v(0,t_h)} \tag{9.4}$$

If the tickler has integer times $t_h = h$, then in (9.4) the unit price $v(0,h)$ can be expressed according to the implicit forward annual rates by (7.30').

Definition (9.4) shows that the duration is a mean of the times on the basis of the economic scenario valued in the reference instant. The h^{th} weight $S_h \cdot v(0,t_h)$ of the mean is the share of present value, or price, at 0 due to supply (t_h,S_h). It is also evident that D as the meaning of the *first moment*. Thus, it is the abscissa of the center of mass regarding the system $\{S_h v(0,t_h)\}$ of mass put on the time axis in the abscissas t_h.

If we assume, in order to obtain valuations, that the flat-yield structure will always be at level i, the duration, in this case named *flat yield curve duration* (*FYC duration*), depending on i or $\delta = \ln(1+i)$, becomes:

$$D = \frac{\sum_{h=1}^{n} t_h S_h (1+i)^{-t_h}}{\sum_{h=1}^{n} S_h (1+i)^{-t_h}} = \frac{\sum_{h=1}^{n} t_h S_h \, e^{-\delta\, t_h}}{\sum_{h=1}^{n} S_h \, e^{-\delta\, t_h}} \tag{9.5}$$

It is easy to prove the following theorem:

Theorem: *For any operation O having an annuity feature,* $\forall i > 0$

$$D \leq z \leq \bar{t} \tag{9.6}$$

results, holding the equalities only if O has only one amount at maturity t_n. *The inequalities are reversed if* $i<0$[2] .

Example 9.1

Let us consider the operation O given by the cash-inflows S_h: {10450, 12500, 8820, 56600} in the times: {1, 2.5, 3.75, 5 }, which are valued using the annual flat-rate i = 4.75%.

Recalling formulae (9.2), (9.3), (9.5), O has the time parameters $\bar{t}$, z , D, defined above. We obtain

1) $\bar{t} = \dfrac{10,450 + 12,500 \cdot 2.5 + 8,820 \cdot 3.75 + 56,600 \cdot 5}{10,450 + 12,500 + 8,820 + 56,600} = \dfrac{357,775}{88,370} = 4,049$;

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2 This theorem, formulated by E. Levi (1964), is proved here in the case of flat-yield structure taking into account known inequalities among means. *Proof*: with only one cash-inflow in t_n , (9.6) is trivial when it gives equalities. With many cash-inflows we firstly prove the strong inequality between $\bar{t}$ and z. Put: $v = 1/(1+i)$, we obtain

$$v^{\bar{t}} = v^{\sum_h t_h S_h / \sum_h S_h} = \left\{ \prod_h \left(v^{t_h} \right)^{S_h} \right\}^{1/\sum_h S_h}$$

Therefore, $v^{\bar{t}}$ is the geometric mean of the discount factors v^{t_h} with weights S_h, then it is smaller than their arithmetic mean with the same weights, which by (9.3) equals v^z. Owing to $v^{\bar{t}} < v^z$, we obtain $z < \bar{t}$ if $i>0$ (that is $v<1$); on the contrary we obtain $z > \bar{t}$ if $i<0$ (that is $v>1$). Moreover we prove the strong inequality between z and D: using $u=1+i$, we obtain

$$u^D = u^{\sum_h t_h S_h v^{t_h} / \sum_h S_h v^{t_h}} = \left\{ \prod_h (u^{t_h})^{S_h v^{t_h}} \right\}^{1/S_h v^{t_h}}$$

Therefore, u^D is the geometric mean of the accumulation factors u^{t_h} with weights S_h , then less than their arithmetic mean with the same weights, which equals u^z, considering the reciprocal in (9.3). Owing to $u^D < u^z$, we obtain $D<z$ if $i>0$ (that is $u>1$); on the contrary we obtain $D>z$ if $i<0$ (that is $u<1$). Finally, by the transitivity of “<” and “>”, $D > \bar{t}$ follows if $i>0$, $D < \bar{t}$ follows if $i<0$ ←.

We can deduce these relations between D and $\bar{t}$ observing that if $i>0$ the discounting of S_h, made on D and not on $\bar{t}$, cause a reduction which is greater for the amounts S_h payable at times nearer to the last maturity, so the weighted arithmetic mean decreases. The opposite conclusion results if $i<0$; in this case we obtain a greater reduction for the payments closer to 0.

2) z is given by:

$88{,}370\cdot 1.0475^{-z} = 10{,}450\cdot 1.0475^{-1} + 12{,}500\cdot 1.0475^{-2.5} +$

$+ 8{,}820\cdot 1.0475^{-3.75} + 56{,}600\cdot 1.0475^{-5}$

that is: $1.0475^{-z} = 73397.46 / 88370 = 0.830570$

$$z = \frac{-\log\ 0.830570}{\log\ 1.0475} = 4{,}000$$

3) the FYC duration D is given by

$$D = \frac{10{,}450\cdot 1.0475^{-1} + 2.5\cdot 125{,}00\cdot 1.0475^{-2.5} + 3.75\cdot 8{,}820\cdot 1.0475^{-3.75} + 5\cdot 56{,}600\cdot 1.0475^{-5}}{10{,}450\cdot 1.0475^{-1} + 12{,}500\cdot 1.0475^{-2.5} + 8{,}820\cdot 1.0475^{-3.75} + 56{,}600\cdot 1.0475^{-5}} =$$

$$= \frac{289{,}991.80}{73{,}397.46} = 3.951$$

We can verify: $\bar{t} \geq Z \geq D$, according to $i > 0$.

Exercise 9.1

With the same cash-inflows virtue as in Example 9.1, let us consider a spot-prices structure $v(0,z) = \dfrac{30}{z+30}$ and calculate z and D.

A. We obtain: $v(0,1) = 0.967742$; $v(0,2.5) = 0.923077$; $v(0,3.75) = 0.888889$; $v(0,5) = 0.857143$

By virtue of (9.3), z is solution to

$$\frac{30}{z+30} = \frac{10{,}450 \cdot 0.967742 + 12{,}500 \cdot 0.923077 + 8{,}820 \cdot 0.888889 + 56{,}600 \cdot 0.857143}{10{,}450 + 12{,}500 + 8{,}820 + 56{,}600} =$$

$$= \frac{78{,}005.66}{88{,}370} = 0.882717 \text{ ; then } \quad z = 3.986$$

By virtue of (9.4), D is given by

$$D = \frac{10{,}450\cdot 0.967742 + 2.5\cdot 12{,}500\cdot 0.923077 + 3.75\cdot 8{,}820\cdot 0.888889 + 5\cdot 56{,}600\cdot 0.857143}{10{,}450\cdot 0.967742 + 12{,}500\cdot 0.923077 + 8{,}820\cdot 0.888889 + 56{,}600\cdot 0.857143} =$$

$$= \frac{310{,}930.53}{78{,}005.66} = 3.986$$

The denominator is the value in 0 of this inflow operation.

For the *duration D* the following property is valid, and is very useful in the subsequent applications:

Let us consider two investments at 0 *in order to obtain the operations* O_1 *and* O_2 *made up respectively of cash-inflows* $\{a_h\}$ *at the maturities* $\{t'_h\}$ *and* $\{b_k\}$ *at* $\{t''_k\}$. *Let us also denote by* $A = \sum_h a_h v(0,t'_h)$ *and* $B = \sum_k b_k v(0,t''_k)$ *the values at* 0 *of* O_1 *and* O_2, *according to the spot prices structure* $v(0,t)$, *or the corresponding rates* $i(0,t)$. *Then the duration* D_{a+b} *of the operation* $O_1 \cup O_2$, *which includes together the cash-inflows of* O_1 *and* O_2 *in the respective maturities, is the arithmetic mean of the duration* D_a *of* O_1 *and* D_b *of* O_2 , *weighted by the values A and B*[3] .

Then the following *mixing property* holds:

Suppose that it is possible to vary continuously and in a proportional way the amounts $\{a_h\}$ *and* $\{b_k\}$ *of two investments which give rise to the operations* O_1 *and* O_2, *so that the values A and B change, but not the durations of* O_1 *and* O_2. *Under this assumption we can continuously vary the shares A/(A+B) and B/(A+B) of two investments so as to obtain a duration of* $O_1 \cup O_2$ *however chosen in the interval between the durations of* O_1 *and* O_2.

The classical case concerns the assignment of the total amount A+B to buy two kinds of securities. A and B are changed as written with $A+B$ = const., so as to obtain the desired duration D_{a+b}. This property can be extended to more than two operations.

In the applications the calculation of the FYC duration is useful for basic operations which are components of a complex portfolio management, when we assume a flat-yield structure and therefore a *FYC duration*. We use this calculation for the following operations.

O = temporary annuity-immediate with constant payments

In order to calculate the *FYC duration*, because of its invariance with respect to proportional variations of amounts, it is not restrictive to consider O as unit annuity. Moreover we assume unit periods and annually delayed payments. By virtue of (9.5) and the symbols in Chapter 5, we obtain

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3 The proof follows the associative feature of the arithmetic mean. Analytically, concerning the duration of $O_1 \cup O_2$ we can be written:

$$D_{a+b} = \frac{\sum_h t'_h\, a_h v(0,t'_h) + \sum_k t''_k\, b_k v(0,t''_k)}{A + B} = D_a \frac{A}{A+B} + D_b \frac{B}{A+B} .$$

$$D = \frac{\sum_{h=1}^{n} h(1+i)^{-h}}{\sum_{h=1}^{n}(1+i)^{-h}} = \frac{(Ia)_{\overline{n}|i}}{a_{\overline{n}|i}} \tag{9.7}$$

where the denominator is the present value $a_{\overline{n}|i} = v\frac{1-v^n}{1-v}$ of the annuity and the numerator is the present value $(Ia)_{\overline{n}|i} = \frac{v}{1-v}[\frac{1-v^n}{1-v} - nv^n]$ of the increasing annuity.[4] We easily obtain the expression of D by i:

$$D = \frac{1+i}{i} - \frac{n}{(1+i)^n - 1} \tag{9.7'}$$

It is easy to verify that the duration given by (9.7') is a decreasing function of the annuity valuation's rate. Moreover, the value $n/[(1+i)^n-1]$ vanishes with diverging n and then the curve $D(n)$ is strictly increasing[5] and bounded by the asymptote $i/(1+i) = 1/d$. This level then gives the FYC duration of a perpetuity.

Example 9.2

Let us consider a semiannual annuity-immediate over 6 years, using the rate of 6.20%. With regard to the duration's calculation, it is equivalent to assume unit payments. Taking the half-year as the unit, we use (9.7') and n=12 half-years and i = 0.030534 (= six-month equivalent rate). The result is

$$D = \frac{1.030534}{0.030534} - \frac{12}{(1.030534)^{12} - 1} = 6.142$$

i.e., FYC duration = 3.071 years = 3y+0m+26d.

O = cash-inflows by zero-coupon bonds (ZCB)

Since the duration is a mean of the cash-inflows times and the ZCB gives only one encashment at maturity n, $D=n$ results. This number is the greatest value obtainable with respect to the durations of bonds with cash-inflows of any amount and period before maturity.

4 See. formulae (5.2) and (5.26) of Chapter 5.

5 To prove the increase of D with n, it is enough to verify that the subtrahend in (9.7') decreases. Indeed, since $(1+i)^{-x} > 1 - x\ln(1+i)$ (= its linear approximation), $\forall x>0$, results, the derivative of $y = x/[(1+i)^{-x} - 1]$ here is negative.

The bonds have a redemption value C and coupon I for the unit period. Then

$$t_h = h,\ (h=1,...,n)\ ;\ S_h = I\ \ (\text{if } h=1,...,n-1),\ S_h = C+I\ \ (\text{if } h=n) \tag{9.8}$$

results, and the FYC duration is obtained taking into account the effect of (9.8) on (9.5). Then we obtain

$$D = \frac{I\ (\mathrm{I}a)_{\overline{n|}i} + n\ C(1+i)^{-n}}{I\ a_{\overline{n|}i} + C(1+i)^{-n}} \tag{9.9}$$

Equation (9.9) can be meaningfully obtained by the mixing property, pointing out that the operation here considered is the union of O' (= cash-inflows of coupons) and O'' (= cash-inflow of redemption principal). The value in 0 of O' is $A=I\cdot a_{\overline{n|}i}$; that of O'' is $B=C(1+i)^{-n}$; the FYC durations are respectively $(\mathrm{I}a)_{\overline{n|}i}/\,a_{\overline{n|}i}$ and n. Calculating their arithmetic mean with weights A and B we obtains (9.9), which is a function decreasing with respect to both the coupon rate I/C and the yield rate i.

In Figure 9.1 the curve of D, as a function of the time, tends to the asymptote $(1+i)/i$. It is strictly increasing only if $I/C \geq i$ (purchase at par or above par); otherwise (purchase below par) it increases up to local maximum $\hat{D} > (1+i)/i$ and then decreases towards the asymptote. However, it is to say that with the customary rates we obtain the local maximum point after a long time, then in the numerical interval of the usual maturities the duration D, as a function of the time t, increases.

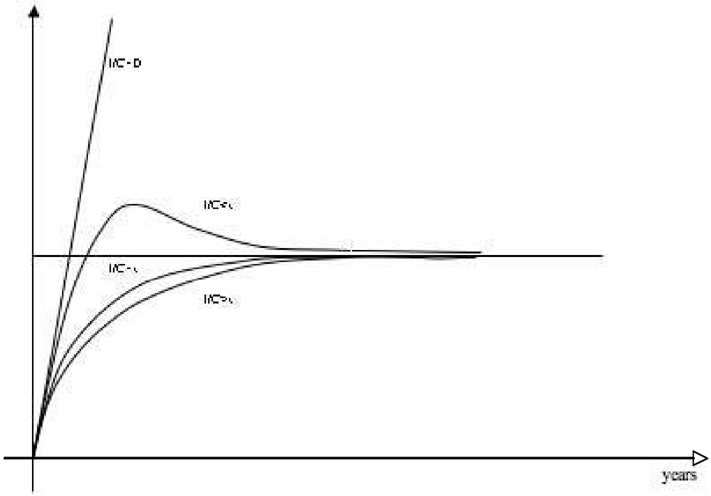

Figure 9.1. *Plot of D, function of the time t*

Example 9.3

Let us consider at t=0 a bond with redemption value 100 in t=5 and annual coupons whose amount is 6.50 payable in 1, 2, 3, 4 and 5. Let us assume the valuation rate = 7%.

The duration's calculation proceeds as follows:

$v = 1.07^{-1} = 0.934579$; n=5 ; C = 100 ; I = 6.50

$$(Ia)_{\overline{n}|i} = \frac{0.934579}{0.065421}\left[\frac{0.287014}{0.065421} - 3.564931\right] = 11.746862$$

$$a_{\overline{n}|i} = \frac{0.287014}{0.07} = 4.100197$$

hence by virtue of (9.9)

$$D = \frac{6.50 \cdot 11.746862 + 5 \cdot 100 \cdot 1.07^{-5}}{6.50 \cdot 4.100197 + 100 \cdot 1.07^{-5}} = 4.419 = 4\text{y} + 5\text{m} + 8\text{d}$$

O = cash-inflows by bond portfolio

The previous calculation for the duration can be extended to the vectorial case, i.e. to a portfolio of m types of bonds whose purchase transfers the rights on m encashment operations, that we assume on the same tickler, e.g. on n years. These cash-inflows in such a tickler can be collected in a matrix $S = \{S_{kh}\}$. Therefore, $O = O_1 \cup ... \cup O_m$ where at any operation

$$O_k = \{S_{k1},...,S_{kn}\}\&\{t_1,...,t_n\},\ k=1,...,m,$$

which concerns a unit of the k^{th} bond, we join the initial value (or purchase price at 0)

$$P_k = \sum_{h=1}^{n} S_{kh}(1+i)^{-t_h};\quad (k = 1,...,m) \tag{9.10}$$

Let us now consider a portfolio obtained by λ_k units of the k^{th} bond. It is evident the cash-inflows due to the given portfolio set up the operation $O = \lambda_1 O_1 \cup ... \cup \lambda_m O_m$. Then the value (or price) P of O at 0 is the linear combination of the values (or prices) P_k of O_k with weights λ_k . In addition, at 0 the FYC duration D of O is the arithmetic mean of D_k, FYC durations of O_k , weighted by the values $\lambda_k P_k$ at 0 of the k^{th} bond's shares in the portfolio[6]. Such conclusions remain valid if,

6 Then it is possible to extend the mixing property for m>2 bonds. For the proof it is sufficient to use the linear algebra. Indeed, the cash-inflows of O in t_h are $P_h = \sum_k \lambda_k S_{kh}$. Then

1) using (9.10) it follows $A = \sum_{h=1}^{n} P_h (1+i)^{-t_h} = \sum_{h=1}^{n}\sum_{k=1}^{m} \lambda_k S_{kh}(1+i)^{-t_h} = \sum_{k=1}^{m} \lambda_k A_k$

instead of a flat-yield curve, we use any discount law (or unit prices structure) $v(0, t)$.

Example 9.4

Let us consider three kinds of bonds, and use 100 as the unit redemption value and 5.50% as valuation flat-rate:

– 1st bond: with constant coupon; maturity 4 years; annual coupon 5;

– 2nd bond: with zero-coupon; maturity 2 years;

– 3rd bond: with variable coupon; maturity 3 years; annual coupons with amounts: 5.40; 5.80; 5.60.

Denoting by λ_k the quantities of the bonds in the portfolio, let us consider two portfolio mix assumptions:

assumption α) $\lambda_1 = 25$; $\lambda_2 = 3$; $\lambda_3 = 10$;

assumption β) $\lambda_1 = 2$; $\lambda_2 = 28$; $\lambda_3 = 8$.

Then, assuming a unit times tickler, the cash-inflows tickler per bond unit and the possible mixing are the following:

$t_h =$	*1*	*2*	*3*	*4*	α	β
1st bond	5	5	5	105	25	2
2nd bond	0	100	0	0	3	28
3rd bond	5.4	5.8	105.6	0	10	8
Total					38	38

We could calculate the FYC duration portfolio by working on the total cash-flows, that in the two given hypotheses are written here below.

$t_h =$	*1*	*2*	*3*	*4*
α	179.0	483.0	1,181.0	2,625.0
β	53.2	2,856.4	854.8	210.0

We obtain

$$D_\alpha = \frac{12530.61630}{3728.320452} = 3.36093 \qquad D_\beta = \frac{8045.04565}{3514.24090} = 2.28927$$

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2) using (9.4') and (9.10), $D_k = \dfrac{\sum_h t_h S_{kh} (1+i)^{-t_h}}{A_k}$; $D = \dfrac{\sum_h t_h P_h (1+i)^{-t_h}}{A} =$

$$= \frac{\sum_h t_h \sum_k \lambda_k S_{kh} (1+i)^{-t_h}}{A} = \sum_k \lambda_k \frac{A_k}{A} \frac{\sum_h t_h S_{kh} (1+i)^{-t_h}}{A_k} = \frac{\sum_k \lambda_k A_k D_k}{\sum_k \lambda_k A_k}$$

However, it is important to calculate the bond unit duration and make the linear combination for each mixing assumption. Denoting by D_s the FYC duration of the s^{th} bond, we easily obtain:

$$D_1 = \frac{365.52882}{98.24742} = 3.72049 \quad ; \quad D_2 = \frac{179.6904}{89.8452} = 2$$
$$D_3 = \frac{285.33174}{100.25991} = 2.84592$$

where in the denominators the values P_k of unit bonds appear. Since the portfolio duration is the arithmetic mean of unit bond durations weighted by the total values of each bond in the portfolio, we obtain

$$D_\alpha = \frac{25 \cdot 98.247424 \cdot 3.720493 + 3 \cdot 89.8452 \cdot 2 + 10 \cdot 100.259910 \cdot 2.845921}{25 \cdot 98.247424 + 3 \cdot 89.8452 + 10 \cdot 100.259910} =$$

$$= \frac{12530.61036}{3728.32030} = 3.36093$$

$$D_\beta = \frac{2 \cdot 98.247424 \cdot 3.720493 + 28 \cdot 89.8452 \cdot 2 + 8 \cdot 100.259910 \cdot 2.845921}{2 \cdot 98.247424 + 28 \cdot 89.8452 + 8 \cdot 100.259910} =$$

$$= \frac{8045.04317}{3514.23973} = 2.28927$$

i.e. we obtain the previous results. At the denominator of D_α and D_β we have the values of the two portfolios α and β, i.e.

$$P^\alpha = \sum_k \lambda_k^\alpha P_k = 3728.32 \quad ; \quad P^\beta = \sum_k \lambda_k^\beta A_k = 3514.24$$

9.2. Variability and dispersion indicators

9.2.1. *2nd order duration*

In the portfolio management it is useful to take into account the dispersion. To satisfy this need, we define the 2nd *order duration* at 0

$$D^{(2)} = \frac{\sum_{h=1}^n t_h^2 S_h v(0,t_h)}{\sum_{h=1}^n S_h v(0,t_h)} \tag{9.11}$$

which has the dimension of (time^2) and depends on the term structure of spot prices $v(0,t_h)$. Equation (9.11) shows that $D^{(2)}$ is the *second moment* of the mass system whose D is the first moment. $D^{(2)} \le (t_n)^2$ always results.

In particular, in the case of a flat-yield structure the *2nd order FYC duration* takes the form of

$$D^{(2)} = \frac{\sum_{h=1}^{n} t_h^2 S_h e^{-\delta t_h}}{\sum_{h=1}^{n} S_h e^{-\delta t_h}} = \frac{\sum_{h=1}^{n} t_h^2 S_h (1+i)^{-t_h}}{\sum_{h=1}^{n} S_h (1+i)^{-t_h}} \qquad (9.11')[7]$$

In addition, it is suitable to look over the consequences of interest rate variability, particularly in the case of investment rate of return (see the immunization theory in section 9.3). By working under a flat-rate, it is known that initial value *V(i)* of a cash-inflow set due to an investment (or the price which allows a rate of return *i*) is a function that decreases and is a downward concave of *i*.

The reference to initial value (or price) $V(\delta)$ and to its derivatives depending on intensity $\delta=\ln(1+i)$ simplifies the following formulae. We obtain

$$V(\delta) = \sum_{h=1}^{n} S_h e^{-\delta\, t_h} \; ; \; V'(\delta) = -\sum_{h=1}^{n} t_h S_h e^{-\delta\, t_h} \; ; \; V''(\delta) = \sum_{h=1}^{n} t_h^2 S_h e^{-\delta\, t_h} \quad (9.12)$$

resulting in: $V(\delta) > 0$; $V'(\delta) < 0$; $V''(\delta) > 0$. [8]

Example 9.5

Let us again use the cash-flow given in Exercise 9.1, i.e. the cash-inflows {10,450; 12,500; 8,820; 56,600} over the tickler {1; 2.5; 3; 3.75; 5}, valued by the law $v(0,z)=30/(z+30)$. We have seen that the value at 0 of the given cash-flow is 78,005.66 and its duration is 3.986 years.

Using some results of that exercise, we verify that the 2nd order FYC duration by virtue of (9.11') is given by

$$D^{(2)} = \frac{10{,}450 \cdot 0.967742 + 2.5^2 \cdot 12{,}500 \cdot 0.923077 + 3.75^2 \cdot 8820 \cdot 0.888889 + 5^2 \cdot 56{,}600 \cdot 0.857143}{10{,}450 \cdot 0.967742 + 12{,}500 \cdot 0.923077 + 8{,}820 \cdot 0.888889 + 56{,}600 \cdot 0.857143} =$$

---

7 From a physical point of view, also with a flat-yield structure the duration *D*, given in this case by (9.5), is the first moment, thus the *center of mass*, of the distribution of the mass $S_h e^{-\delta\, t_h}$ put in t_h, whereas $D^{(2)}$ given by (9.11') is the second moment, that is the *moment of inertia* in a rotation around the origin. Moreover $\sigma^2 = D^{(2)} - D^2$ is the *variance,* i.e. the central second moment (or central moment of inertia), which is a *dispersion indicator*. In a more general approach with any term structure, the mass $S_h v(0,t)$ are taken, *D* is given by (9.4) and $D^{(2)}$ is given by (9.11) being valid analogous conclusions.

8 It is well known that the sign of second derivative measures, if this sign is positive, the punctual degree of upward concavity (or downward convexity) of a *f(x)* and, if this sign is negative, that of downward concavity (or upward convexity). The concavity and the convexity imply “downward”.

$$= \frac{1,405,335.65}{78,005.66} = 18.0158 \ \text{years}^2$$

9.2.2. *Relative variation*

Let us carry out a survey of variability indicators under the flat-yield structure. With reference to the function $V(\delta)$ and its first derivative (see (9.12)), we can define an index of *relative variation* by

$$\frac{V'(\delta)}{V(\delta)} = \frac{d}{d\delta} \ln V(\delta) < 0 . \tag{9.13}$$

Recalling (9.5), which gives the FYC duration D, the basic formula

$$V'(\delta)/V(\delta) = - D \tag{9.13'}$$

that identifies in absolute value the quickness of relative variation of V with respect to δ, with the FYC duration, holds.[9]

Note

Among the consequences of rate fluctuations there is also that of the same duration change, which in previous approximations is neglected. Under a flat-yield structure the quickness and the direction of such a variation are measured by the derivative of D. Using (9.12) this results in:

$$\frac{\partial D}{\partial \delta} = \frac{\partial}{\partial \delta} \frac{\sum_{h=1}^{n} t_h S_h e^{-\delta t_h}}{\sum_{h=1}^{n} S_h e^{-\delta t_h}} = \frac{-\sum_{h=1}^{n} t_h^2 S_h e^{-\delta t_h} . V(\delta) + (\sum_{h=1}^{n} t_h S_h e^{-\delta t_h})^2}{V^2(\delta)} =$$

$$-[D^{(2)} - D^2] = -\sigma^2 < 0 \tag{9.14}$$

Therefore $\partial D / \partial \delta$ is a meaningful *volatility indicator* of times with respect to mean time D. By virtue of (9.14) it follows that D decreases when intensity or rate increases. We obtain the following equation

$$\frac{\partial D}{\partial i} = \frac{\partial D}{\partial \delta} \frac{d\delta}{di} = - v \, \sigma^2 < 0 \tag{9.14'}$$

9 A type of duality holds between duration and interest instantaneous intensity: intensity (=time^{-1}) is the derivative of value's logarithm (pure number because it is an exponent) with respect to time; duration (=time) is the derivative of value's logarithm (pure number because it is an exponent) with respect to intensity (time^{-1}).

To conclude: distribution variance ⇒ quickness of $D(\delta)$ variation ⇒ quickness of $D(i)$ variation.

9.2.3. *Elasticity*

In the flat-yield structure assumption, we define *elasticity* η_δ of a bond value (or price) at 0 with respect to δ[10] the limit ratio on vanishing $\Delta\delta$ between the relative variations $\Delta V/V$ and $\Delta\delta/\delta$. The result is

$$\eta_\delta = \lim_{\Delta\delta \to 0} \frac{\Delta V/V}{\Delta\delta/\delta} = \delta \frac{V'(\delta)}{V(\delta)} = -\delta D \qquad (9.15)$$

Denoting by η_i the elasticity with respect to $i = e^\delta - 1$, the result is:

$$\eta_i = \lim_{\Delta i \to 0} \frac{\Delta V/V}{\Delta i/i} = i \frac{V'(i)}{V(i)} = -\frac{i}{1+i} D \qquad (9.15')$$

9.2.4. *Convexity and volatility convexity*

Under the flat-yield structure assumption, let us introduce two further indicators linked to second derivative (>0) of value (or price) V. The former indicator, called *convexity*, is the level of convexity per unit of value. The convexity can be expressed as a function of the intensity δ, called δ-*convexity* and denoted by γ_δ, as well as by a function of rate i, called i-*convexity* and denoted by γ_i. Due to (9.11'), the δ-*convexity* coincides with the *2nd order FYC duration*. Using symbols, the two indicators valued at 0 are:

$$\gamma_\delta = \frac{\sum_{h=1}^n t_h^2 S_h e^{-\delta t_h}}{\sum_{h=1}^n S_h e^{-\delta t_h}} = D^{(2)} = \frac{V''(\delta)}{V(\delta)} \qquad (9.16)$$

$$\gamma_i = \frac{\sum_{h=1}^n t_h (t_h+1) S_h (1+i)^{-t_h}}{\sum_{h=1}^n S_h (1+i)^{-t_h}} = \frac{V''(i)}{V(i)}(1+i)^2 \qquad (9.16')$$

---

10 In general, given two variables x, y functionally linked by $y=f(x)$ (continuous and derivable), we define *elasticity* of y with respect to x , here denoted by η, the punctual relative increment of y with respect to x, that is the limit ratio between their relative variations. Using symbols

$$\eta = \lim_{\Delta x \to 0} \frac{\{f(x+\Delta x) - f(x)\}/f(x)}{\Delta x/x} = \frac{x}{f(x)} \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = x\frac{f'(x)}{f(x)} = \frac{d[\ln f(x)]}{d(\ln x)}$$

The latter indicator, called *volatility-convexity*, means the convexity per unit of value variation. The *volatility-convexity* can be expressed according to the intensity δ, called δ-*volatility-convexity* and denoted by γ_δ^*, or dependent on rate *i*, called *i-volatility-convexity* and denoted by γ_i^*. Using symbols:

$$\gamma_\delta^* = -\frac{\sum_{h=1}^{n} t_h^2 S_h e^{-\delta\, t_h}}{\sum_{h=1}^{n} t_h S_h e^{-\delta\, t_h}} = -\frac{D^{(2)}}{D} = \frac{V''(\delta)}{V'(\delta)} \qquad (9.17)$$

$$\gamma_i^* = -\frac{\sum_{h=1}^{n} t_h (t_h+1) S_h (1+i)^{-t_h}}{\sum_{h=1}^{n} t_h S_h (1+i)^{-t_h}} = \gamma_\delta^* - 1 = \frac{V''(i)}{V'(i)}(1+i) \qquad (9.17)$$

Comparing (9.5) with (9.16) and (9.16') we obtain the important simple formula: $\gamma_i = \gamma_\delta + D$, which enables us to easily calculate one of the quantities having been given the others. In addition, such indicators are applied in the theory of classical immunization, which we address in section 9.3.

Exercise 9.2

Given the inflows operation *J* with amounts [8,520; 11,400; 6,450; 61,800] and tickler [0.5; 2; 3.5; 5.25], due to a previous investment with amount calculable by (5.23), let us calculate the duration, the convexity and the volatility-convexity at 0, with respect to δ and *i*, valuing by *i* = 4.75% or by the corresponding δ.

A. Using an *Excel* spreadsheet, we draw up the following table which gives the asked solutions by working on the data of *J*. The constraints among *i*-convexity, δ-convexity and FYC duration are verified.

CALCULUS OF DURATION, CONVEXITY AND VOLATILITY-CONVEXITY

	Depending on δ		$\delta =$	0.046406	
t_h	S_h	$v_h=\exp(-\delta t_h)$	$S_h v_h$	$t_h S_h v_h$	$t_h^2 S_h v_h$
0.50	8,520	0.977064	8,324.58	4,162.29	2,081.15
2.00	11,400	0.911364	10,389.55	20,779.10	41,558.20
3.50	6,450	0.850082	5,483.03	19,190.60	67,167.11
5.25	61,800	0.783775	48,437.29	254,295.76	1,335,052.72
Σ			72,634.45	298,427.76	1,445,859.19
	$V =$	72634.45		$D =$	4.1086
	$\gamma_\delta =$	19.9060		$\gamma^*_\delta =$	-4.8449

	Depending on i:		$i = e^\delta - 1 =$	0.047500	
t_h	S_h	$v_h = (1+i)^{-t_h}$	$S_h v_h$	$t_h S_h v_h$	$t_h(t_h+1) S_h v_h$
0.50	8,520	0.977064	8,324.58	4,162.29	6243.44
2.00	11,400	0.911364	10,389.55	20,779.10	62337.31
3.50	6,450	0.850082	5,483.03	19,190.60	86357.72
5.25	61,800	0.783775	48,437.29	298,427.76	1589348.48
Σ			72,634.45	298,427.76	1,744,286.94
	$V =$	72634.45		$D =$	4.1086
	$\gamma_i =$	24.0146		$\gamma^*_I =$	-5.8449
			The constraint	$\gamma_i = \gamma_\delta + D$	is verified

Table 9.1. *Example of calculus of duration, convexity and volatility-convexity*

The *Excel* instructions are the following. With regard to non-empty cells, we have:

E14: input of annual rate; E3:= ln(1+E4).

Depending on δ :

from row 5 to 8:

column A: maturity: input from A5 to A8;
column B: flow: input from B5 to B8;
column C: unit spot price: C5:= EXP(-E3*A5); copy C5, then paste on C6 to C8
column D: present value: D5:= B5*C5); copy D5, then paste on D6 to D8;
column E: present value · maturity: E5:= A5*D5); copy E5, then paste on E6 to E8;
column F: present value · maturity^2: F5:= A5*E5); copy F5, then paste on F6 to F8;
row 9: sums: D9:= SUM(D5:D8); copy D9, then paste on E9 to F9;
row 11: value, duration: C11:= D9 ; F11:= E9/D9;

row 12: δ-convexity, δ-volatility-convexity: C12:= F9/D9 ; F12:= -F9/E9;

Depending on i:

from row 16 to 19:

column A: maturity: copy from A5 to A8, then paste on A16 to A19;
column B: flow: copy from B5 to B8, then paste on B16 to B19;
column C: unit spot price: C16:= (1+E$14)^-A16; copy C16, then paste on C17 to C19;
column D: present value: D16:= B16*C16); copy D16, then paste on D17 to D19;
column E: present value · maturity: E16:= A16*D16); copy E16, then paste on E17 to E19;
column F: present values · maturity · (maturity+1): F16:= E16*(A16+1); copyF16, then paste on F17 to F19;
row 20: sums: D20:= SUM(D16:D19); copy D20, then paste on E20 to F20;
row 22: value, duration: C22:= D20 ; F22:= E20/D20;
row 23: *i*-convexity, *i*-volatility-convexity: C23:= F20/D20 ; F23:= -F20/E20.

9.2.5. *Approximated estimations of price fluctuation*

Let us explain, using the assumption of a flat-yield structure, an alternative interpretation of FYC duration and convexity. Multiplying by a small enough spread $d\delta$ we obtain the approximate formula:

$$\frac{\Delta V(\delta)}{V(\delta)} \cong \frac{V'(\delta)}{V(\delta)} d\delta = -D\, d\delta \tag{9.18}$$

which gives a significant sense of FYC duration. Indeed, since $\Delta V(\delta)/V(\delta)$ gives the rate of $V(\delta)$ variation, by multiplying D by a small increase (or small decrease) of δ, we obtain in an approximate way the corresponding relative decrease (or relative increase) of $V(\delta)$.[11] For this reason D is a *1st order sensitivity indicator* of price with respect to rate changes. By virtue of (9.18), we deduce the simple formula

$$V(\delta_0 + d\delta) \cong V(\delta_0)(1 - D\, d\delta) \tag{9.18'}$$

obtained by the Taylor expansion, restricted to the 1st order, over $V(\delta)$. It allows an approximate estimate of new price consequent to a market rate change in regard to bond, whose price and duration are given according to a previous rate.

In addition, let us observe that the convexity is a *2nd order sensitiveness indicator* of price with respect to rate changes. Along with duration, it enables us to improve the rough valuation of variation of values (or prices) depending on the variation of

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11 Therefore, with the same change of δ, in a bond having high (or low) duration, we obtain a high (or low) relative change of price, having an opposite sign with respect to that of $d\delta$. Thus, this rule follows: it is better to invest in bonds with low duration in case of expectation of increasing rates; on the contrary, to invest in bonds with high duration in case of expectation of decreasing rates.

market intensity, expanding the Taylor formula $V(\delta)$ up to 2nd order. Then we obtain the following improved estimate:

$$V(\delta_0 + d\delta) \cong V(\delta_0) + V'(\delta_0)d\delta + V''(\delta_0)(d\delta)^2/2 =$$

$$= V(\delta_0)(1 - D\, d\delta + \gamma_\delta(d\delta)^2/2) \tag{9.19}$$

and then the consequent relative variation

$$\frac{\Delta V(\delta)}{V(\delta)} \cong -D\; d\delta + \gamma_\delta (d\delta)^2/2 \tag{9.19'}$$

Example 9.6

Let us consider at 0 a bond that gives rise to the distribution of J specified in Exercise 9.2. Under the annual rate i_0 = 4.75% or the corresponding intensity δ_0 = 0.049406, the values D = 4.1086; γ_δ = 19.9060 have been obtained. Let us calculate by an *Excel* spreadsheet, given below, the value (or price) at 0 corresponding to δ_0 and the values (or prices) at 0 corresponding to spreads $d\delta$ = +0.003 and $d\delta$ = -0.004.

CALCULUS OF BOND PRICES BY DURATION (given δ)

Duration =	4.1086		δ-convexity =	19.9060
	Intensity δ =	0.046406	0.049406	0.042406
Amounts	*Maturities*	*Values at* 0	*Values at* 0	*Values at* 0
8.520.00	0.50	8,324.59	8,312.11	8,341.25
11.400.00	2.00	10,389.56	10,327.41	10,473.01
6.450.00	3.50	5,483.04	5,425.77	5,560.34
61.800.00	5.25	48,437.38	47,680.47	49,465.32
True initial price	=	72,634.56	71,745.75	73,839.92
Initial price using (9.18')	=	72,634.56	71,739.28	73,828.27
Initial price using (9.19)	=	72,634.56	71,745.79	73,839.84
True ΔV/V =		0.000000	-0.012237	0.016595
Approximate ΔV/V using (9.18) =		0.000000	-0.012326	0.016434
Approximate ΔV/V using (9.19') =		0.000000	-0.012236	0.016594

Table 9.2. *Example of calculus of bond prices*

In this table, after data inputs (duration and three intensities) the subsequent four rows give by column the amounts, the maturities and the inflow present value depending on the three intensities. Then in the following rows the prices at 0 are calculated by adding up, by column, and, for each intensity, are compared with their estimates according to (9.18') and (9.19). The subsequent three rows give comparisons among the relative variations of true prices and those deduced by (9.18) and (9.19').

The *Excel* instructions for non-empty cells are as follows: duration in B3 and three intensities in C4, C5, C6. Rows from 7 to 10:

column A:	inflow data;
column B:	time data;
columns C,D,E	(cash-inflows present values): C7:= $A7*EXP(-C$4*$B7); copy C7, then paste on C8-C10, on D7-D10, on E7-E10;
row 12:	C12:= SUM(C7:C11); copy C12, then paste on D12-E12;
row 13:	C13:= $C12*(1-$B3*(C4-$C4)); copy C13, then paste on D13-E13;
row 14:	C14:= $C12*(1-$B3*(C4-$C4)+$E3*(C4-$C4)^2/2); copy C14, then paste on D14-E14;
row 16:	C16:= C12/$C12-1; copy C16, then paste on D16-E16;
row 17:	C17:= -$B3*(C4-$C4); copy C17, then paste on D17-E17;
row 18:	C18:= -$B3*(C4-$C4)+$E3*(C4-$C4)^2/2; copyC18, then paste on D18-E18.

Let us now reconsider the previous expansions, assuming the rate i to be a variable of yield (let us recall (9.16') and (9.17')). In this case, taking into account the formulae

$$V(i) = \sum_{h=1}^{n} S_h (1+i)^{-t_h}, \quad V'(i) = -\sum_{h=1}^{n} t_h S_h (1+i)^{-t_h - 1}$$

we immediately obtain:

$$\frac{V'(i)}{V(i)} = \frac{-D}{1+i} = -D\,v \tag{9.20}$$

that also follows from (9.13') by observing that $\frac{d\delta}{di} = \frac{d\ln(1+i)}{di} = v$ and then $\frac{1}{V}\frac{dV}{di} = \frac{1}{V}\frac{dV}{d\delta}\frac{d\delta}{di} = \frac{-D}{1+i}$. Therefore, to make the previous approximations with use of the annual rate, the same expansions can be repeated using $D^*=D/(1+i)$ (called *modified duration* or *volatility*) instead of D. In particular, (9.18) becomes

$$\frac{\Delta V(i)}{V(i)} \cong \frac{V'(i)}{V(i)}di = d\ln V(i) = \frac{-D}{1+i}di \tag{9.20'}$$

and (9.18') becomes

$$V(i_0+di)=V(i_0)\left[1-\frac{D}{1+i_0}\ di\right] \qquad (9.20'')$$

Also, for *V(i)* we can find a better approximation of its change estimate by also considering (9.16') and Taylor expansion up to 2nd order. Thus, we obtain a better estimate by

$$V(i_0+di)\cong V(i_0)+V'(i_0)di+V''(i_0)(di)^2/2=$$

$$=V(i_0)\left[1-\frac{D}{1+i_0}\ di+\frac{\gamma_\delta}{2(1+i_0)^2}(di)^2\right] \qquad (9.21)$$

and by (9.21) the consequent relative variation depending on *i*:

$$\frac{\Delta V(i)}{V(i)}\cong-\frac{D}{1+i}\ di+\frac{\gamma_i}{2(1+i)^2}\gamma_\delta(di)^2 \qquad (9.21')$$

Example 9.7

Let us again take Example 9.6 with the same cash-inflow distribution, but considering rate variations. Under the annual rate i_0 = 4.75% we obtained in Exercise 9.2 the following values: D = 4.1086; γ_i = 24,0146. Let us now calculate, using *Excel* table below, the value (or price) at 0 corresponding to i_0 and the values (or prices) at 0 corresponding to rate variations di = 0.004 and di = -0,004 as well as the relative variations.

CALCULUS OF BOND PRICES BY DURATION (given *i*)				
Duration =	4.1086		*i*-convexity=	24.0146
		Rate *i* = 0.0475	0.0515	0.0435
Amounts	*Maturities*	*Values at* 0	*Values at* 0	*Values at* 0
8,520.00	0.50	8,324.58	8,308.74	8,340.52
11,400.00	2.00	10,389.55	10,310.66	10,469.36
6,450.00	3.50	5,483.03	5,410.37	5,556.95
61,800.00	5.25	48,437.29	47,477.71	49,420.04
True initial price =		72,634.45	71,507.48	73,786.87
Initial price using (9.20") =		72,634.45	71,494.88	73,774.03
Initial price using (9.21) =		72,634.45	71,507.60	73,786.74
True ΔV/V =		0.000000	-0.015516	0.015866
Approximate ΔV/V using (9.20') =		0.000000	-0.015689	0.015689
Approximate ΔV/V using (9.21') =		0.000000	-0.015514	0.015864

Table 9.3. *Example of calculus of bond prices*

The *Excel* instructions are as follows. Duration in B3 and the three rates in C4, C5, C6. Rows 7 to 10:

column A: inflow data;
column B: time data;
columns C,D,E (inflows present values): C7:= $A7*(1+C$4)^-$B7; copy C7, then paste on C8-C10, on D7-D10, on E7-E10;
row 12: C12:= SUM(C7:C11); copy C12, then paste on D12-E12;
row 13: C13:= $C12*(1-$B3*(C4-$C4)/(1+$C4)); copy C15, then paste on D15-E15;
row 14: C14:=$C12*(1-$B3*(C4--$C4)/(1+$C4)++$E3*(C4$C4)^2/(2*(1+$C4)^2)); copy C14, then paste on D14-E14;
row 16: C16:= C12/$C12-1; copy C16, then paste on D16-E16;
row 17: C17:= -$B3*(C4-$C4)/(1+$C4); copy C17, then paste on D17-E17;
row 18: C18:=-$B3*(C4-$C4)/(1+$C4)+$E3*(C4--$C4)^2/(2*(1+$C4)^2); copy C18, then paste on D18-E18.

A generalization

We can analyze the change of value (or price) in more general assumptions, using the symbols in (7.28) and assuming a spot-price structure

$$\{v(0,h)\} = \{[1+i_h]^{-h}\}.$$

For the sake of simplicity we consider a bond implying cash-inflow due to varying coupons I_h and redemption in C. The price (or value) at 0 of such a bond is given by

$$V = \sum_{h=1}^{n} I_h (1+i_h)^{-h} + C(1+i_n)^{-n} \qquad (9.22)$$

The duration D at 0 on the basis of this structure by virtue of (9.4) is

$$D = \left\{\sum_{h=1}^{n} h\ I_h (1+i_h)^{-h} + n\ C(1+i_n)^{-n}\right\} / V\ . \qquad (9.23)$$

V can be considered a function of spot-rates $i_1, i_2, ..., i_n$. Its total differential, corresponding to increments of spot-rates all equal to Δ, is

$$dV = -\left\{\sum_{h=1}^{n} h\ I_h (1+i_h)^{-h-1} + n\ C(1+i_n)^{-n-1}\right\} \Delta = -D^{*} V\ \Delta \qquad (9.24)$$

depending on a *modified duration* D^*, that here is equal to

$$D^{*} = \left\{\sum_{h=1}^{n} h\ I_h (1+i_h)^{-h-1} + n\ C(1+i_n)^{-n-1}\right\} / V\ . \qquad (9.23')$$

By dividing the sides of (9.24) by V, we obtain the relative variation

$$\frac{dV}{V} = -D^{*}\Delta \qquad (9.25)$$

that generalizes (9.13') and highlights that D^* is a sensitivity index. From (9.25) we find that

$$V(i_1 + \Delta, ..., i_n + \Delta) = V(i_1, ..., i_n)(1 - D^{*}\Delta) \qquad (9.25')$$

which generalizes (9.20'') and easily gives the new price corresponding to a uniform variation of rate structure.

9.3. Rate risk and classical immunization

9.3.1. *An introduction to financial risk*

Among the more frequently discussed problems concerning risk theory in finance are those of *interest rate risk*. Such a risk also appears in operations agreed under certainty and considered safe from risks, such as the investments in bonds. To clarify the problems of the risk theory we refer only to investments in bonds, bearing in mind that the application's field is much wider.

As shown in sections 6.9 and 6.10, in a bond loan where all the securities have the same maturity (and we talk about only one maturity) the *rate of return* (IRR) is defined as that rate at which is zero the present value, calculated at the issue, of the algebraic sum of the cash-flow owing to the buyer of the bonds. In case of differentiated maturities, e.g. by a draw rule, the *ex-ante yield* is a mean value in relation to the redemption maturities of the bonds. We define the bond *ex-post rate* the as the real rate achieved according to the date of refund and then to the realized length of life. We saw that the ex-post rates always coincide with the coupon rate for the bonds whose purchase value coincides with the par and redemption value (i.e., *par bonds*).

Examining this more closely, because in a financial operation's valuation it is necessary to take into account all the payments made in the time horizon of such an operation, then referring to only the coupon bond (or more than one coupon bond, but where all the bonds have the same maturity) it is necessary to distinguish three types of yield:

a) the *initial yield*, i.e. the IRR, also called the *ex-ante rate* and denoted by r_i , which is the rate that makes the present value (at the moment issue or purchase) of both receipts and payments equal. Then r_i is obtained not considering the reinvestment of coupons cashed during the bond's lifetime, or else considering them, but – as it will soon be proved – supposing that the reinvestments are profitable according to a rate equal to IRR (then supposing that the curve of the market rates is *flat-yield curve* throughout the bond's lifetime). Moreover, this rate coincides with the *yield rate* defined in section 7.2 in the case of bonds with a certain return and constant coupon or ZCB;

b) the *yield at maturity*, here denoted by r_m , i.e. an ex-post rate realized on a bond at its maturity, taking into account the reinvestment rates obtained on the cashed coupons;

c) the *yield in advance*[12], here denoted by r_a , which is analogous to r_m but referred to a sale and realization before the *maturity*.

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12 Obviously the yield in advance has not to mistake for the discount rate (or advance interest rate) defined in Chapter 3.

Let us prove the equivalence stated in a) and summarized as the following:

Theorem A. Let us suppose that issue (or purchase) price at 0, nominal value and redemption value of a bond are equal to *C*, so that $r_i = i$ (= coupon rate). If in bond management we also consider the reinvestments of coupons as cashed up to maturity and their yield is r_i , then $r_m = r_i$ holds true. On the contrary, without reinvestments, $r_m < r_i$ holds true.

Proof. The latter point is evident after proving the former one. For this purpose we observe that each of *n* coupons is equal to $R = Ci$. Let *F(n)* be the accumulated value of cash-inflows. Using the given assumptions and with *C* as the redemption value, we obtain

$$F(n) = C + R + R(1+r_i) + ... + R(1+r_i)^{n-1} = C + R\frac{(1+r_i)^n - 1}{r_i}$$

In addition, with *C* as the purchase price and r_i as the coupon rate, then $R = Cr_i$,

$$(1+r_m)^n = \frac{C + R\left[(1+r_i)^n - 1\right]/r_i}{C} = 1 + r_i\frac{(1+r_i)^n - 1}{r_i} = (1+r_i)^n$$

results. Thus $r_m = r_i$. ←

In light of the previous reasoning, it is evident that the bondholder must have to consider as random the return of reinvestment revenue due to future cashed coupons as well as the bond price in the case of future sale before the fixed maturity, which is calculated by discounting, at the time of sale, the future flows due to the buyer as coupons and redemption. Hence the *financial rate risk*, which is of two types:

1) *reinvestment risk*, which is the due to the future random fluctuation of market rate on the reinvestment of cashed coupons;

2) *realization risk*, which is the due to the future random fluctuation of the same market rate on the bond price in case of sale in advance.

The effects of two risks are not in accordance with each other; then we obtain a partial compensation, whose degree depends on sale time $t'' \in [0,n]$, where $[0,n]$ is the time interval of investment.

Let us explain the problem with reference to an investment operation *O* in $[0,n]$ with the only outcome being $-P$ at 0 and receipts being $R_h>0$ at time $t_h \in [0,n]$ where $t_n = n$. Such quantities enable the valuation, at 0, of the rate of return r_i. Let *r(t)* be the rate of return, generally varying with respect to the time. It is evident that $r(0) = r_i$.

In the ideal assumption that the market rate be invariant in the whole interval $[0,n]$, the yield of O retains the level of the rate $r(0)$, since at such a rate we can reinvest the intermediate revenues R_h[13]. In case of selling in advance, the transferor's and transferee's returns depend on the transfer price. However, if at this price the seller retains the rate of return x, such a rate is also valid for the buyer[14].

However, if $\exists\, t > 0$ such that $r(t) \neq r_i$, then owing to market rate variation regarding reinvestments and price of realization in advance, the performances change and a decrease is possible, and the expectations, which were valued at purchase time, fail. Then the problem of immunizing arises, i.e. of neutralizing the effects of risk due to rate *r(t)* fluctuations.

Limiting ourselves to operation $J = \{t_h\}$ & $\{R_h\}$ of inflows, regarding its value $V(t,r)$ at t, subject to $(t_k \le t \le t_{k+1})$ under rate r, the result is: $V(t,r)= F(t,r) + P(t,r)$, where

$$F(t,r) = \sum_{h=1}^{k} R_h (1+r)^{t-t_h} \qquad (9.26)$$

is the *accumulated amount* at t, on reinvesting under rate r the cash-inflows before t, and

$$P(t,r) = \sum_{h=k+1}^{n} R_h (1+r)^{-(t_h-t)} \qquad (9.26')$$

is the *present value* at t under rate r of cash-inflows after t, then the *price of realization in advance* at t. Obviously this results in

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13 Let us use as an example a bond as specified in section 6.10, bought in 0 at the price z (so generalizing the previous theorem) with c as the redemption at time n and annual coupons according to the rate i. By the defining equation, whose solution is the (initial) yield rate x, then written as: $-z+ci\ a_{\overline{n}|x} +c(1+x)^{-n} = 0$, we obtain, multiplying by $(1+x)^n$: $ci\ s_{\overline{n}|x} +c = z(1+x)^n$. The left side is the economic outcome in n of z invested in 0, with reinvestments according to the rate x of coupons as cashed. Since it equals the right side $z(1+x)^n$, the *ex-post* yield is x. The opposite is also true.

14 Referring to the bond in footnote 13, in case of a sale after only m years with price p, and of coupon reinvestment at rate x both by the seller and by the buyer, the fairness equation of O on x, quoted in footnote 13, can be written (multiplying by $(1+x)^m$ and considering that, if $n>m$, $a_{\overline{n}|x} = a_{\overline{m}|x} +(1+x)^{-m} a_{\overline{n-m}|x}$), as:

$$[-z(1+x)^m + ci\ s_{\overline{m}|x} +p] +\{-p + ci\ a_{\overline{n-m}|x} + c(1+x)^{-(n-m)}\}$$

The F quantity in square brackets is the value in m of the transferor's O' operation, whereas the P quantity in curly parentheses is the value in m of the transferee's O'' operation. If p is such that F=0, i.e. it is the retrospective reserve in m, O' is fair under rate x; then x is the transferor's rate of return. However, because of the O fairness the price p is also the prospective reserve in m, then P=0 and then O'' is fair under rate x; then x is also the transferee's rate of return.

$$V(t,r) = F(t,r) + P(t,r) = \sum_{h=1}^{n} R_h (1+r)^{t-t_h} \tag{9.27}$$

Given t, it is evident (and immediately verified, using the derivative with respect to r) that $F(t,r)$, obtained by accumulating, is an increasing function of r, whereas $P(t,r)$, obtained by discounting, is a decreasing function of r.

Let us assume, for the sake of simplicity, that in $[0,n]$ the function $r(t)$ is subject to only one variation in $t' \le t_1$, changing from $r(0)$ to $r* = r(0)+ \Delta r$ (where $\Delta r>0$ or $\Delta r<0$). Under such a change, assuming $t_1 \le t \le t_n$, if t is close to t_1 , the variation of F is small whereas that, opposite in sign, of P is large. Then by virtue of (9.27) the V variation has the sign of the P variation. On the contrary, if t is close to t_n , the variation of F is large whereas that, opposite in sign, of P is small. Then due to (9.27) the V variation has the sign of the F variation. Owing to the continuity of such functions, this result implies the existence of a *critical time* $\hat{t}$ regarding the sale in advance, which produces opposite values of F and P variations. Then V remains unchanged. Using symbols we have: $V(\hat{t}, r*) = V(\hat{t}, r(0))$. Thus, we obtain a thorough neutralization of $r(t)$ variation's effects on such values, then on r_a rate, which would agree with $r_i = r_m$ without following the variations of the initial market rate $r(0)$. The calculation of such a critical time is based on classic immunization theory, which will be addressed in section 9.3.2.

The following examples, which recall an exercise given in Devolder (1993), refer to different settings of realization time t'' from that of market rate change (assumed to be only one) t' and the maturity n of a bond with annual coupons; for simplicity they all refer to the purchase of a security at issue (at 0) with purchase price = par value = redemption value = 100, then $r_i = r(0)$ = coupon rate.

Example 9.8. Sale in advance at time $t''=t'=2$ *of a bond with maturity* $n=10$.

Let us put $r(0) = r_i = 0.05 = 5\%$ and assume that the set Ω of "states", concerning the dynamics of the market rate $r(t)$ into the interval [0,10], is given only by the following events:

ω_0= (no change of $r(t)$ at $t \in [0,10]$);

ω_1= (only one change of $r(t)$ at $t_0 = 2$, given by $\Delta = +0.01 = +1\%$);

ω_2= (only one change of $r(t)$ at $t_0 = 2$, given by $\Delta = -0.01 = -1\%$);

Clearly, if ω_0 is true, it results in $r_a = r_m = r_i = 0.05$. Let us consider two other events ω_1 and ω_2, denoting by (ω) the dependence on the Ω state.

The sum $F_\omega(2)$, accrued by an investor owing to cashed coupons at periodic maturities and reinvested up to sale at $t''=2$, do not depend on the Ω state, because changes of $r(t)$ into [0,2) do not occur. The sum is given by

$$F_\omega(2) = 5\ (1.05) + 5 = 10.25$$

The sale price $P_\omega(2)$ follows by rates *r(t)* in [2,10], thus depends on the Ω state:

$$P_\omega(2) = 5a_{\overline{8}|r(\omega)} + 100[1 + r(\omega)]^{-8}$$

– if $\omega = \omega_1$: $r(\omega_1) = 0.06$, $P_{\omega_1}(2) = 31.05 + 62.74 = 93.79$

– if $\omega = \omega_2$: $r(\omega_2) = 0.04$, $P_{\omega_2}(2) = 33.66 + 73.07 = 106.73$

The seller's total revenue at $t'' = 2$ is $S_\omega(2) = F_\omega(2) + P_\omega(2)$. Then

– if $\omega = \omega_1$: $r(\omega_1) = 0{,}06$, $S_{\omega_1}(2) = 104.04$

– if $\omega = \omega_2$: $r(\omega_2) = 0{,}04$, $S_{\omega_2}(2) = 116.98$

The yield in advance $r_a(\omega)$ depends on Ω state, as it is solution of

$$100\ [1 + r_a(\omega)]^2 = S_\omega(2)$$

If $\omega = \omega_1$, we obtain: $r_a(\omega_1) = 0.020000$; if $\omega = \omega_2$: $r_a(\omega_2) = 0.081573$, then $r_a(\omega_1) < r_a(\omega_2)$ with a large difference among them and r_i which is in the middle. As $t'' = t'$, a reinvestment risk does not exist, because the coupons are reinvested in [0,2] under certain rate $r(0) = 0.05$ whereas the risk of realization exists with a large decrease (increase) of the sale price and of the yield in advance when the market rate increases (decreases).

*Example 9.9. Sale in advance of a bond with maturity n=*10 *at time t''=*6 *in the middle from t' and n.*

On the basis of the data and events set out in Example 9.8, except for $t''=6$, we obtain the following results.

The sum $F_\omega(6)$, accrued by the investor due to cashed coupons at periodic maturities and reinvested up to sale at $t'' = 6$, depends on the Ω state and is given by

$$F_\omega(6) = 5\ \{1.05\ [1 + r(\omega)]^4 + s_{\overline{5}|r(\omega)}\}$$

– if $\omega = \omega_1$: $r(\omega_1) = 0.06$, $F_{\omega_1}(6) = 5\ (1.325601 + 5.637093) = 34.81$;

– if $\omega = \omega_2$: $r(\omega_2) = 0.04$, $F_{\omega_2}(6) = 5\ (1.228351 + 5.416323) = 33.22$.

The sale price $P_\omega(6)$ depends on the Ω state and is given by

$$P_\omega(6) = 5\ a_{\overline{4}|r(\omega)} + 100\ [1 + r(\omega)]^4$$

– if $\omega = \omega_1$: $r(\omega_1) = 0.06$, $P_{\omega_1}(6) = 17.32 + 79.21 = 96.53$;

– if $\omega = \omega_2$: $r(\omega_2) = 0.04$, $P_{\omega_2}(6) = 18.15 + 85.48 = 103.63$.

The seller's total revenue at $t'' = 6$ is $S_\omega(6) = F_\omega(6) + P_\omega(6)$. Then

- if $\omega = \omega_1$: $r(\omega_1) = 0.06$, $S_{\omega_1}(6) = 131.34$;
- if $\omega = \omega_2$: $r(\omega_2) = 0.04$, $S_{\omega_2}(6) = 136.85$.

The yield in advance $r_a(\omega)$ depends on the Ω state, as it is the solution of

$$100\,[1 + r_a(\omega)]^6 = S_\omega(6)$$

If $\omega = \omega_1$, we obtain: $r_a(\omega_1) = 0.046485$; if $\omega = \omega_2$: $r_a(\omega_2) = 0.053677$.

Compared to the results of Example 9.8, the difference between $r_a(\omega_1)$ and $r_a(\omega_2)$ is much reduced, since these rates are approaching the value of the initial market rate, 0.05. As $t'<t''<n$, both the reinvestment risk on cashed coupons from time 2 to 6, and the realization risk exist, owing to the advance of the sale in respect to the maturity, which implies a discount from time 10 to 6 under a random market rate.

Example 9.10. Realization of a bond at maturity n=10

On the basis of the data and events set out in Example 9.8, except for $t''=10$, we obtain the following results.

The sum $F_\omega(10)$, accrued by the investor due to cashed coupons at periodic maturities and reinvested up to realization at time 10, depends on the Ω state and is given by

$$F_\omega(10) = 5\ \{(1.05)[1 + r(\omega)]^8 + s_{\overline{9}|r(\omega)}\}$$

- if $\omega = \omega_1$: $r(\omega_1) = 0.06$, $F_{\omega_1}(10) = 5\,(1.673540+11.491316) = 65.82$;
- if $\omega = \omega_2$: $r(\omega_2) = 0.04$, $F_{\omega_2}(10) = 5\,(1.368569+10.582795) = 59.76$.

The realization value is certainly $P_\omega(10)=100$; it does not depend on the Ω state, as it lacks a discount under a random rate.

The seller's total revenue at $t''=10$ is $S_\omega(10) = F_\omega(10) + P_\omega(10)$. Then

- if $\omega = \omega_1$: $r(\omega_1) = 0.06$, $S_{\omega_1}(10) = 165.82$;
- if $\omega = \omega_2$: $r(\omega_2) = 0.04$, $S_{\omega_2}(10) = 159.76$.

The yield in advance $r_a(\omega)$ becomes yield to maturity $r_m(\omega)$ because the realization occurs at fixed maturity; it depends on Ω state, as it is the solution of

$$100\,[1 + r_m(\omega)]^{10} = S_\omega(10)$$

If $\omega = \omega_1$, we obtain: $r_m(\omega_1) = 0.051874$; if $\omega = \omega_2$: $r_m(\omega_2) = 0.047965$ then $r_m(\omega_1) > r_m(\omega_2)$ with a small difference between them and r_i which is in the middle. As $t'' = n$, a realization risk does not exists but the reinvestment risk exists with an increase (decrease) of total revenue and yield at maturity when the market rate increases (decreases).

Note

In Examples 9.8 to 9.10 when $2=t' \leq t''<n=10$, the rates of return in the middle, between those achieved for $t''=2$ and $t''=10$, have been obtained. By varying t'' continuously from the time 2 to 10, the rate $r_a(\omega_1)$ increases from 0.0200 to 0.0519, whereas the rate $r_a(\omega_2)$ decreases from 0.0816 to 0.0480. Then it is plausible that, as $r_a(\omega_1)$ and $r_a(\omega_2)$ are continuous functions of t'', we can settle on a *critical time* $\hat{t}$ of investment ($2<\hat{t}<10$) for which $r_a(\omega_1)=r_a(\omega_2)$, so that two opposite effects of a market rate's change exactly compensate one another. Then, for this critical time $\hat{t}$ we obtains:

$$r_a(\omega_1) = r_a(\omega_2) = r_a(\omega_0) = 0.05 = r_i \quad \text{(certain rate).}$$

In such a way the risk rate is removed.

9.3.2. *Preliminaries to classic immunization*

In section 9.3.1 we dealt with rate risk and critical time $\hat{t}$ of investment, which allows the removal of such a risk by suitable methods. Now we address processes, called *classic immunization*, that we also call *semi-deterministic* because all elements of involved operations are fixed except for the market interest rate, which is exposed to random changes.

We will begin with the critical time calculation which removes risk rate in a particular context. We will give some theorems concerning semi-deterministic immunization, distinguishing between problem of *cover of single liability* and *cover of multiple liabilities* problems[15].

The market term structure, if not flat-yield, will be identified by temporal changes of intensity $\delta(x,u)$ as defined in Chapter 2, where x is the time of agreement or valuation and u is the current time (see section 7.5.3 for other characteristic quantities of term structure).

In classic immunization we usually take the hypothesis of *additive shifts* of rates, i.e. of random changes Y_k, from x to t, of the instantaneous intensities corresponding to them, whose result is $Z(x,t)=\sum_k Y_k$. Therefore, with $x<t<y$

15 For a thorough analysis on such subjects, see Devolder (1993) and De Felice Moriconi (1991).

$$\delta(t,y) = \delta(x,y) + Z(x,t) \qquad (9.28)^{16}$$

However, for simplicity we will proceed under the assumption of only additive shifts in the considered time interval.

9.3.3. *The optimal time of realization*

In section 9.3.1 we have seen that, in the case of only random additive shifts, for continuity in the interval of financial cash-inflows operation J a critical time $\hat{t}$ exists, such that the random change of value (and thus of the fulfilled rate of return) due to additive shifts, vanishes. Now we look for the calculation of this $\hat{t}$.

It is not restrictive, and it simplifies symbols, to put the time origin in the instant of J valuation and of rate (or intensity $\delta(0,u)$) agreement. Moreover, let us assume that in the J interval only one additive shift on $\delta(0,u)$ of random size Y occurs in the market at time t', before times $\{t_k\}$ (k=1,...,n), set in chronological order, where the inflows of J, components of vector $a = \{a_k\}$, are cashed. Thus, the intensity $\delta(0,u)$ from 0 to t' and $\delta(t',u)$ from t' to t_n are in force in the market, linked by

$$\delta(t',u) = \delta(0,u) + Y \ , \ 0 < t' < t_1 < ... < t_k < ... < t_n \ ; u > t' \qquad (9.29)$$

Let us denote by $V(T,\underline{a};Y)$ (where the 3rd variable represents the size of a possible shift) the value in $T \le t_n$ of total revenue due to $\underline{a}$, obtained adding reinvestment revenue and realization revenue. Thus, this value depends on random shift size. Lacking shift, it results in

$$V(T,\underline{a};0) = \sum_{k=1}^{n} a_k e^{-\int_T^{t_k} \delta(0,u)du} \qquad (9.30)$$

On the other hand, if the additive shift Y occurs at $t' < t_1$, according to (9.29) the total revenue due to $\underline{a}$ at T is given (by distinguishing reinvestment and realization components) by

$$V(T,\underline{a};Y) = \sum_{k:t_k \le T} a_k e^{\int_{t_k}^{T} \delta(t',u)du} + \sum_{k:t_k > T} a_k e^{-\int_T^{t_k} \delta(t',u)du} =$$

$$= \sum_{k=1}^{n} a_k e^{-\int_T^{t_k} \delta(t',u)du} = \sum_{k=1}^{n} a_k e^{-\int_T^{t_k} \delta(0,u)du} e^{-Y(t_k - T)}$$

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16 In the case of flat-yield structure, unless additive shifts, (9.28) becomes: $\delta_t = \delta_x + Z(x,t)$, where δ_u is the intensity agreed at u.

Thus

$$V(T,\underline{a};Y)=\frac{1}{v(0,T)}\sum_{k=1}^{n}a_k v(0,t_k)e^{-Y(t_k-T)} \qquad (9.30')$$

where $v(0,t)=e^{-\int_0^t \delta(0,u)du}$ is the price at 0 of an unitary zero coupon bond (UZCB) having maturity at t, valued according to $\delta(0,u)$ (see (7.42)).

Since the second derivative with respect to Y of $V(T,\underline{a};Y)$ for every Y is positive, the function $f(Y)=V(T,\underline{a};Y)$ has the absolute minimum point at Y=0 (then $V(T,\underline{a};Y)\geq V(T,\underline{a};0)$ for every Y if the first derivative of $f(Y)$ vanishes at 0. Then we obtain the immunization. However, this sentence is true if T is chosen equal to the duration of J. In fact, due to

$$\left[\frac{\partial}{\partial Y}V(T,\underline{a};Y)\right]_{Y=0}=\frac{1}{v(0,T)}\sum_{k=1}^{n}a_k v(0,t_k)=0$$

it follows that

$$T=\frac{\sum_{k=1}^{n}t_k a_k v(0,t_k)}{\sum_{k=1}^{n}a_k v(0,t_k)}=D_J(0)$$

Then we conclude: $\hat{t}=D_J(0)$, i.e., *the critical time for immunizing against interest rate risk is the duration of J valued at* 0. Moreover, $\hat{t}$ is the only solution to the problem.

Example 9.11

Carrying out Examples 9.8, 9.9 and 9.10, on the basis of data and events specified in Example 9.8, except for t'', let us verify that, putting the investment time equal to duration, we obtain immunization.

Let us buy the bond at 0 and redeem it at par in a maturity of 10 years, par value 100, rate $r(0)=r_i=0.05=5\%$. The duration at 0, according to (9.9), is worth D = 8.107822. Let us calculate the economic results obtainable under the various states of Ω.

$$F_\omega(8.107822)=5\,\{(1.05)[1+r(\omega)]^6+s_{\overline{7}|r(\omega)}\}[1+r(\omega)]^{0.107822}$$

- if $\omega=\omega_1$: $Y=+0.01$, $r(\omega_1)=0.06$, $F_{\omega_1}(8.107822)=49.73$;
- if $\omega=\omega_2$: $Y=-0.01$, $r(\omega_2)=0.04$, $F_{\omega_2}(8.107822)=46.33$;

$$P_\omega(8.107822) = \{5\ a_{\overline{2}|r(\omega)} + 100\,[1+r(\omega)]^{-2}\}[1+r(\omega)]^{0.107822}$$

- if $\omega = \omega_1$: $Y = +0.01$, $r((\omega_1) = 0.06$, $P_{\omega_1}(8.107822) = 98.79$
- if $\omega = \omega_2$: $Y = -0.01$, $r((\omega_2) = 0.04$, $P_{\omega_2}(8.107822) = 102.32$;

The seller's total revenue at $t_1 = 8.107822$ is

$S_\omega(8.107822) = F_\omega(8.107822) + P_\omega(8.107822)$. Then

- if $\omega = \omega_1$: $Y = +0.01$, $r((\omega_1) = 0.06$, $S_{\omega_1}(8.107822) = 148.52$;
- if $\omega = \omega_2$: $Y = -0.01$, $r((\omega_2) = 0.04$, $S_{\omega_2}(8.107822) = 148.65$.

The yield in advance $r_a(\omega)$ depends on state, as it is the solution of

$100\,[1+r_a(\omega)]^{8.107822} = S_\omega(8.107822)$

If $\omega = \omega_1$, we obtain: $r_a(\omega_1) = 0.0500$; if $\omega = \omega_2$: $r_a(\omega_2) = 0.0501$

To conclude: $S_{\omega_1}(8.107822) \cong S_{\omega_2}(8.107822)$ and $r_a(\omega_1) \cong r_a(\omega_2) \cong 0.05$. Therefore, we obtain immunization against rate risk using an investment the time length of which is its duration = 8.107822.

9.3.4. *The meaning of classical immunization*

Let us proceed, step by step, to analyze in depth the immunization with respect to yield shifts under increasing generalization, summarizing the characteristic features of a theory which would need a wider treatment.

For the sake of simplicity, let us use 0 for the valuation time where the intensity $\delta(0,u)$ identifying the structure is agreed. We refer to operation O giving a vector $\underline{a} = (a_1,..., a_n)$ of cash-inflows (also called *assets*) and a vector $\underline{b} = (b_1,..., b_n)$ of cash-outflows (also called *liabilities*). It is not restrictive to assume that $\underline{a}$ and $\underline{b}$ have the same tickler $\underline{t} = (t_1,..., t_n)$, under the constraints $\{a_h \geq 0\}$,$\{b_h \geq 0\}$, because $\underline{t}$ can be obtained by the union of $\{a_h > 0\}$ and $\{b_h > 0\}$ ticklers[17]. Denoting by $V(0,\underline{a};0)$ the value at 0 of assets and by $V(0,\underline{b};0)$ that of liabilities, if $V(0,\underline{a};0) = V(0,\underline{b};0)$ results, we can tell that the flows $\underline{a}$ and $\underline{b}$ are *in equilibrium*. This equality is also called a *budget constraint*. Moreover, by definition flows $\underline{a}$ and $\underline{b}$ are immunized if, with only one additive shift Y (positive or negative, and with small size) at the time $t' < t_1 < ... < t_n$, $V(0,\underline{a};Y) \geq V(0,\underline{b};Y)$ holds. This weak inequality assures *the cover by* $\underline{a}$

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17 In such a case, if compensations between assets and liabilities are allowed, then at each maturity t_h we cannot have net receipts a_h-b_h and net outlays b_h-a_h both positive.

of the liabilities $\underline{b}$[18]. Denoting by $\underline{s} = (s_1,..., s_n)$, where $s_h = a_h - b_h$, the *net flows* vector and by $V(0,\underline{s};0)$ its value at 0, the equilibrium implies: $V(0,\underline{s};0) = 0$ and we have immunization if furthermore $V(0,\underline{s};Y) \geq 0$. In other words, immunization implies that the function $f(Y) = V(0,\underline{s};Y)$ has a local minimum point at $Y = 0$.

9.3.5. *Single liability cover*

We have immunization against random additive shift following the Fisher-Weil theorem (1971) if the revenue due to a "portfolio" at the end of the period of its management is, in case of an additive shift, not lower than that obtainable without a shift. It is easy to prove that to keep the bond up to maturity, on reinvesting the encashments, generally does not give immunization (see Example 9.10).

Let us state the version of the Fisher-Weil theorem that works on present values and gives the immunization conditions in asset portfolio management to cover only one liability (or, which is the same, a financial target which implies future outlays) under any term structure.

Theorem B (Fisher-Weil). Given the intensity $\delta(0,u)$ summarizing the structure at 0, let b be the amount of a payment scheduled at time $T>0$ and $\underline{a} = (a_1,..., a_n)$ be an asset flow at positive times $t_1<...<t_n$. Assume the value at 0 of $\underline{a}$ is equal to that of b according to $\delta(0,u)$, i.e., the following budget constraint is valid:

$$V(0,\underline{a};0) = V(0,b;0) \tag{9.31}$$

If at t', where $0<t'<t_1$, a random additive shift Y according to (9.29) occurs, then for the values calculated under the new intensity

$$V(t',\underline{a};Y) \geq V(t',b;Y) \tag{9.32}$$

results, if and only if the duration of $\underline{a}$ calculated at 0 equals maturity T of the liability.

Proof. Using

$$\rho(\underline{a},b;0) = V(0,\underline{a};0)/V(0,b;0) =$$

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18 It would be more convenient to use $t' = 0$ for an immediate comparison with the equilibrium case. However, this is not needed. We can observe that

$$V(0,\underline{a};Y) = e^{-\int_0^{t'}\delta(0,u)du} V(t',\underline{a};Y),\ V(0,\underline{b};Y) = e^{-\int_0^{t'}\delta(0,u)du} V(t',\underline{b};Y);$$

then $V(t',\underline{a};Y) \geq V(t',\underline{b};Y)$ implies $V(0,\underline{a};Y) \geq V(0,\underline{b};Y)$, and vice versa. It must be highlighted that in the times following t' the discounts have carried out using the intensity $\delta(t',u)$.

$$= \frac{\sum_{k=1}^{n} a_k e^{-\int_0^{t_k}\delta(0,u)du}}{b\, e^{-\int_0^T \delta(0,u)du}} = \frac{1}{b}\sum_{k=1}^{n} a_k e^{\int_{t_k}^{T}\delta(0,u)du} \qquad (9.33)$$

because of (9.31) ρ(a,b;0) =1 results. After shift Y at t', ρ(a,b;0) is modified in

$$g(Y) = \rho(\underline{a},b;t',Y) = V(t',\underline{a};Y)/V(t',b;Y) = \frac{\sum_{k=1}^{n} a_k e^{-\int_0^{t'}\delta(0,u)du}\, e^{-\int_{t'}^{t_k}\delta(t',u)du}}{b\, e^{-\int_0^{t'}\delta(0,u)du}\, e^{-\int_{t'}^{T}\delta(t',u)du}} \qquad (9.34)$$

thus, due to (9.29)

$$g(Y) = \rho(\underline{a},b;t',Y) = \frac{1}{b}\sum_{k=1}^{n} a_k e^{\int_{t_k}^{T}\delta(0,u)du}\, e^{Y(T-t_k)} \qquad (9.34')$$

By calculating the first and second derivative of $g(Y)$ we obtain

$$g'(Y) = \frac{1}{b}\sum_{k=1}^{n}(T-t_k)a_k e^{\int_{t_k}^{T}\delta(0,u)du}\, e^{Y(T-t_k)} \qquad (9.35)$$

$$g''(Y) = \frac{1}{b}\sum_{k=1}^{n}(T-t_k)^2 a_k e^{\int_{t_k}^{T}\delta(0,u)du}\, e^{Y(T-t_k)} \qquad (9.36)$$

We obtain: $g''(Y) > 0$, $\forall Y$, then (9.34') is a convex function. If and only if $g'(0)=0$, $g(Y)$ holds the minimum point at Y=0 where its value is 1. Therefore, around Y=0 it results in $g(Y) = \rho(\underline{a},b;t',Y) \geq 1$, i.e. (9.32) holds. However, owing to (7.42) and (9.35), $g'(0)=0$ is equivalent to

$$\frac{\sum_{k=1}^{n}(T-t_k)a_k v(0,t_k)}{b\, v(0,T)} = 0$$

Taking into account the budget constraint in (9.31), written as $\sum_{k=1}^{n} a_k v(0,t_k) = b\, v(0,T)$, the equation $g'(0)=0$ is also equivalent to

$$D := \frac{\sum_{k=1}^{n} t_k a_k v(0,t_k)}{\sum_{k=1}^{n} a_k v(0,t_k)} = T$$

Summarizing the reasoning, the budget constraint in (9.31) between $\underline{a}$ and $\underline{b}$ signifies, if the term rates structure remains unchanged, the suitability of receipts $\underline{a}$

under the tickler $\underline{t} = \{t_1, ..., t_n\}$, $0<t_1<...<t_n$, for covering outlay (or target) b at time T, accumulating or discounting by law $v(0,t)$. Under a random additive shift, the cover is still assured provided that T equals $D_0(\underline{a})$, i.e. the duration of $\underline{a}$ at 0[19]. Immunization gives a guarantee of yield at the minimum assured rate $\{b/V(0,\underline{a};0)-1\}$. Theorem B can be applied to the selection of immune portfolios in order to obtain a *single liability cover.*

The operational meaning of Theorem B is as follows. To obtain immunization, we should build a portfolio of assets, the duration of which in 0 equals T. This is always possible, because of the duration's mixing property (see section 9.1.4) and the associative property of the averages considered here (see section 2.5.2).

In fact, let us assume that in 0 the market gives two bond packages (that without loss of generality we can assume to be of the ZCB type). Let each bond of such packages be the redemption values U_1 and U_2 at maturities t_1 and t_2, $(t_1<T<t_2)$, respectively. If $T=t_1$ or $T=t_2$ occurs, the immunization problem would be trivially solved, choosing only one of the packages. The market financial law should be identified by spot prices $\{v(0,u)\}$, $(0\leq u\leq t_2)$. We can settle the portfolio $\underline{a} = (a_1, a_2)$ with tickler $\underline{t} = (t_1, t_2)$ to cover the liability b (or to assure the target b) in T, by calculating the shares (i.e. the numbers α_1, α_2 of the bonds of two packages) to make up $\underline{a}$ so as to satisfy the budget constraint on values at 0 and the constraints on $\underline{a}$ duration at 0. Using $V(0,b) = b\, v(0,T)$, is sufficient to solve the linear system

$$\begin{cases} \alpha_1 U_1 v(0,t_1) + \alpha_2 U_2 v(0,t_2) = V(0,b;0) \\ t_1\alpha_1 U_1 v(0,t_1) + t_2\alpha_2 U_2 v(0,t_2) = T\, V(0,b;0) \end{cases} \qquad (9.37)$$

If linear independence between such equations holds, we obtain the following only solution

$$\alpha_1 = \frac{V(0,b;0)(t_2-T)}{U_1 v(0,t_1)(t_2-t_1)} \ , \ \alpha_2 = \frac{V(0,b;0)(T-t_1)}{U_2 v(0,t_2)(t_2-t_1)} \qquad (9.38)$$

If N types of ZCB subject to law $\{v(0,u)\}$ are available in the market, having par values $U_1, U_2, ..., U_n$, is sufficient to put them into two subgroups and, owing to the mixing property, to obtain two portfolios having face value amounts U_1^*, U_2^* and durations t_1, t_2 to substitute into (9.37)[20].

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19 This condition can also be written as equality between T-t' and the duration $D_{t'}(\underline{a})$ valued at t'. In fact, the duration is a mean of the times and, denoting by $D_{t'}$ and D_0 the durations calculated in t' and in 0, we obtain: $D_{t'} = D_0 - t'$.

20 If bonds are not ZCB, we consider that each coupon bond is equivalent to a group of ZCB, the face value of which equals the coupons or the redemption value.

Exercise 9.3

Let us use two types of ZCB, called $\mathcal{A}$ and $\mathcal{B}$: $\mathcal{A}$ has redemption values \$1,000 at maturity 6; $\mathcal{B}$ has redemption value \$500 at maturity 9. Let us calculate the numerical shares of $\mathcal{A}$ and $\mathcal{B}$ to obtain the cover of \$98,000 at time 7.25 (=7y+3m) if the financial market law is settled by intensity $\delta(0.u)$=0.06-0.002u. Let us verify the immunization by examples.

A. According to given data, we obtain:

$U_1 = 1000$; $U_2 = 500$; $t_1 = 6$; $t_2 = 9$; $T = 7.25$; $v(0,u) = e^{-\int_0^u [0.06-0.002z]dz}$ and then $v(0,6) = e^{-0.324} = 0.723250$; $v(0,9) = e^{-0.459} = 0.631915$; $v(0,7.25) = e^{-0.382} = 0.682197$.

Applying (9.38) we obtain

$$\alpha_1 = \frac{98000 \bullet 0.682197 \bullet 1.75}{1000 \bullet 0.723250 \bullet 3} = 53.921726 \cong 54$$

$$\alpha_2 = \frac{98000 \bullet 0.682197 \bullet 1.25}{500 \bullet 0.631915 \bullet 3} = 88.164856 \cong 88$$

Let us verify the budget constraint in terms of present values at 0.

On 1st bond:	53.921716·1000·0.723250	=	38,998.89
On 2nd bond:	88.164856·500·0.631915	=	27,956.36
Asset present value		=	66,855.25
Liability present value	98,000·0.682197	=	66,855.25

Let us assume that at time 5 a random additive shift occurs with the following possible events

$\Delta = +0.01$ i.e. $\delta_{+}(5,u) = 0.07 - 0.002\,u$

$\Delta = -0{,}01$ i.e. $\delta_{-}(5,u) = 0.05 - 0.002\,u$

Thus, the new spot prices at 0 are:

if $\Delta = +0.01$:

$$v^{+}(0,6) = e^{-\int_0^5 [0.06-0.002z]dz} e^{-\int_5^6 [0.07-0.002z]dz} = 0.759572 \bullet 0.942707 = 0.716054$$

$$v^{+}(0,9) = e^{-\int_0^5 [0.06-0.002z]dz} e^{-\int_5^9 [0.07-0.002z]dz} = 0.759572 \bullet 0.799315 = 0.607137$$

$v^{+}(0,7.25) =$

$$= e^{-\int_0^5 [0.06-0.002z]dz} e^{-\int_5^{7.25} [0.07-0.002z]dz} = 0.759572 \bullet 0.878150 = 0.667018;$$

if Δ = -0.01:

$$v^{-}(0,6) = e^{-\int_0^5 [0.06-0.002z]dz} e^{-\int_5^6 [0.05-0.002z]dz} = 0.759572 \bullet 0.961751 = 0.730519$$

$$v^{-}(0,9) = e^{-\int_0^5 [0.06-0.002z]dz} e^{-\int_5^9 [0.05-0.002z]dz} = 0.759572 \bullet 0.865888 = 0.657704$$

$$v^{-}(0,7.25) =$$

$$= e^{-\int_0^5 [0,06-0,002z]dz} e^{-\int_5^{7,25} [0,05-0,002z]dz} = 0.759572 \bullet 0.918569 = 0.697719$$

Let us verify immunization with respect to given additive shifts:

if Δ = +0.01:		
on 1st bond:	53,921716·1000·0,716054	= 38,610.86
on 2nd bond:	88,164856·500·0,607137	= 26,764.07
present value of assets		= 65,374.93
present value of liabilities	98000·0,667018	= 65,367.76
If Δ = -0,01:		
on 1st bond:	53,921716·1000·0,730519	= 39,390.84
on 2nd bond:	88,164856·500·0,657704	= 28,993.19
present value of assets		= 68,384.03
present value of liabilities	98,000·0.697719	= 68,376.46

If both Δ = +0,01 and Δ = -0,01: asset present value ≥ liability present value.

9.3.6. *Multiple liability cover*

The immunization problem with regard to single liability cover can be generalized into that of multiple liabilities cover, i.e. with reference to many outlays (or financial obligations). Then we assumes that the operator must deal to pay many debts $\underline{b}$ (*liabilities*), spread over time, by means of many receipts due to credits $\underline{a}$ (*assets*). Such a process is called: *Asset-Liability Management* (ALM).

Let us consider an initial balance statement in terms of the present value of assets $\underline{a} = (a_1,..., a_n)$, $a_h \geq 0$, and of liabilities $\underline{b} = (b_1,..., b_n)$, $b_h \geq 0$, according to the market rate in force at time 0. $\underline{t} = (t_1,..., t_n)$ $(0<t_1< ...<t_n)$ is the common[21] tickler of $\underline{a}$ and $\underline{b}$. However, under what conditions does the initial equilibrium not change into unfavorable imbalance under a subsequent change of the market rates' structure?

21 As already seen, this coincidence is not restrictive if we refer to the union of $\underline{a}>\underline{0}$ and $\underline{b}>\underline{0}$ ticklers.

It is evident that an easy solution is obtained using an asset portfolio “devoted” to a given liabilities vector, that is: $a_h = b_h$, $\forall t_h$. In such a case each receipt corresponds to an outlay with the same amount and maturity. Then the former exactly covers the latter without residual debts or credits. However, such a situation, i.e. a sufficient condition of immunization, is quite unusual.

For situations when equality does not occur between distributions of cash inflows and outflows, a rule, with regard to the rate risk of insurance companies under flat-yield-curve hypothesis for the market rates, was first given by Redington (1952). Bearing in mind Redington’s rule, let us assume a balance statement at 0 between assets and liabilities, without shift, given by a budget constraint

$$V(0,\underline{a};0) = \sum_{k=1}^{n} a_k e^{-\delta t_k} = \sum_{k=1}^{n} b_k e^{-\delta t_k} = V(0,\underline{b};0) \qquad (9.39)$$

where $V(0,\underline{a};0)$ and $V(0,\underline{b};0)$ are the values of $\underline{a}$ and $\underline{b}$ at 0 without shift and δ is the intensity in force at time 0. Still denoting by $\underline{s} = (s_1,..., s_n)$, where $s_h = a_h - b_h$, the vector of net flows, (9.39) is equivalent to $V(0,\underline{s};0)=0$, which means the fairness of the whole operation the valued according δ. If an additive shift occurs, the following theorem holds

Theorem C (Redington). Let us assume that at 0 the constant intensity δ and (9.39) holds in the market and that an additive shift from δ to $\delta+Y$, with random $|Y|$ sufficiently small occurs just after 0[22]. Thus, according to previous definitions about $\underline{a}$, $\underline{b}$, $\underline{t}$, a sufficient condition to realize immunization, i.e.

$$V(0,\underline{s};Y) = V(0,\underline{a};Y) - V(0,\underline{b};Y) \geq 0 \qquad (9.40)$$

– where the values at 0 are calculated in the hypothesis of shift Y – is that both

$$\sum_{k=1}^{n} t_k a_k e^{-\delta t_k} = \sum_{k=1}^{n} t_k b_k e^{-\delta t_k} \qquad (9.41)$$

and

$$\sum_{k=1}^{n} t_k^2 a_k e^{-\delta t_k} > \sum_{k=1}^{n} t_k^2 b_k e^{-\delta t_k} \qquad (9.42)$$

hold.

Proof. Equation (9.41) signifies equality between the first derivatives of $\underline{a}$ and $\underline{b}$ in Y=0, i.e. $V'(0,\underline{a};0) = V'(0,\underline{b};0)$. Equation (9.42) signifies inequality between their second derivatives in Y=0, i.e. $V''(0,\underline{a};0) > V''(0,\underline{b};0)$. This implies that

$$V'(0,\underline{s};0) = 0 \ ; \quad V''(0,\underline{s};0) > 0. \qquad (9.43)$$

22 This specification, given for the sake of simplicity, is not basic: these results also hold with a shift in some time after 0 but before t_1.

The truth of the system in (9.43) is, as well known, a sufficient condition in order that $V(0,\underline{s};Y)=\sum_{k=1}^{n} s_k e^{-(\delta+Y)t_k}$ has a relative minimum into Y=0, so with $|Y|$ sufficiently small, (9.40) holds[23].

Recalling (9.13') and (9.16) and taking into account the budget constraint in (9.39), we can observe that (9.41) leads to equality

$$D(\underline{a})=D(\underline{b}) \tag{9.41'}$$

which is the well known necessary Redington condition for immunization with regard to ALM. Moreover, still owing to the budget constraint, (9.42) leads to inequality

$$\gamma_\delta(\underline{a})>\gamma_\delta(\underline{b}) \tag{9.42'}$$

Therefore, the immunization condition for multiple liability cover can meaningfully be formulated requiring that at time 0 the duration of assets are equal to that of liabilities and the convexity of assets are larger than that of liabilities (inequality satisfied, of course, in case of single liability cover and in the Fisher-Weil theorem).

Under the two hypotheses of budget constraint and equality of durations, the inequality condition in (9.42') between asset and liabiliy convexities implies the following meaning of immunization: a market rate decrease (a market rate increase) leads to an increase (a decrease) of the value of the assets which is larger (smaller) than that of the liabilities. Then in both shift cases we obtain a net margin increase.

We must still observe that (9.42') implies

$$\sigma^2(\underline{a})>\sigma^2(\underline{b}) \tag{9.42''}$$

where $\sigma^2(\underline{a})$ and $\sigma^2(\underline{b})$ are the variances of $\underline{a}$ and $\underline{b}$, i.e. the central second moments of distributions ($\underline{t}$&$\underline{a}$) and ($\underline{t}$&$\underline{b}$). To prove this statement, it is sufficient to recall the equalities $\sigma^2=D^2-D^{(2)}$ and equation (9.41').

Both observations can be generalized to the case of variable rates under a term structure and possible additive shifts. In relation to this argument let us now give a theorem generalizing the Redington condition under financial law following $\delta(x,u)$ intensity.

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23 It is evident proof of Theorem C can be obtained by the Taylor expansion up to 2nd order of $V(0,\underline{s};Y)$ with starting point Y=0.

Theorem D (generalization of Redington theorem). Let $\delta(0,u)$ be the intensity current at time 0 on the market. Given two cash-flows, the former with assets $\underline{a}$ = $(a_1,..., a_n)$, $(a_h \geq 0)$, the latter with liabilities $\underline{b} = (b_1,..., b_n)$, $(b_h \geq 0)$, both with tickler $\underline{t}$ = $(t_1,..., t_n)$, $0<t_1< ...<t_n$. Let us assume that $\underline{a}$ and $\underline{b}$ are balanced under $\delta(0,u)$, or that the budget constraint

$$V(0,\underline{a};0) = \sum_{k=1}^{n} a_k e^{-\int_0^{t_k} \delta(0,u)du} = \sum_{k=1}^{n} b_k e^{-\int_0^{t_k} \delta(0,u)du} = V(0,\underline{b};0) \quad (9.44)$$

holds. In addition, we suppose that $\delta(0,u)$ has at t' just after 0[24] an additive infinitesimal shift Y according to (9.29) with $t' = 0^+$ for simplicity. Then

$$V(t',\underline{a};Y) \geq V(t',\underline{b};Y) \quad (9.45)$$

(equivalent to $V(0,\underline{a};Y) \geq V(0,\underline{b};Y)$ and implying immunization against shift Y) holds, if, valuing with the use of $\delta(0,u)$, equality (9.41') between $\underline{a}$ and $\underline{b}$ durations at 0, i.e.

$$\sum_{k=1}^{n} t_k a_k e^{-\int_0^{t_k} \delta(0,u)du} / V(0,\underline{a};0) = \sum_{k=1}^{n} t_k b_k e^{-\int_0^{t_k} \delta(0,u)du} / V(0,b;0),$$

is verified, as well as the inequality

$$D^{(2)}(\underline{a}) > D^{(2)}(\underline{b}) \quad (9.46)$$

between $\underline{a}$ and $\underline{b}$ 2nd order durations in 0, i.e.

$$\sum_{k=1}^{n} t_k^2 a_k e^{-\int_0^{t_k} \delta(0,u)du} / V(0,\underline{a};0) > \sum_{k=1}^{n} t_k^2 b_k e^{-\int_0^{t_k} \delta(0,u)du} / V(0,\underline{a};0)$$

is valid.

Proof. With reference to net amounts $\underline{s} = \underline{a} - \underline{b}$, let us denote by

$$D(\underline{s}) = \sum_{k=1}^{n} t_k s_k e^{-\int_0^{t_k} \delta(0,u)du} / V(0,\underline{s};0)$$

the $\underline{s}$ duration in 0. We obtain:

$$D(\underline{s}) = D(\underline{a}) - D(\underline{b}) = \left(\frac{\partial}{\partial Y} V(0,\underline{s};Y) \right)_{Y=0} .$$

In addition, let us denote by

$$D^{(2)}(\underline{s}) = \sum_{k=1}^{n} t_k^2 s_k e^{-\int_0^{t_k} \delta(0,u)du} / V(0,\underline{s};0)$$

24 See also footnote 22.

the $\underline{s}$ 2nd order duration, resulting in:

$$D^{(2)}(\underline{s}) = D^{(2)}(\underline{a}) - D^{(2)}(\underline{b}) = \left(\frac{\partial^2}{\partial Y^2} V(0,\underline{s};Y)\right)_{Y=0}$$

Let us consider the Taylor expansion of $V(0, \underline{s}; Y)$, starting by Y=0, up to 1st order and using the 2nd order remainder. We obtain, with η included between 0 and Y,

$$V(0,\underline{s};Y) = V(0,\underline{s};0) + \left(\frac{\partial}{\partial Y} V(0,\underline{s};Y)\right)_{Y=0} Y + \frac{1}{2}\left(\frac{\partial^2}{\partial Y^2} V(0,\underline{s};Y)\right)_{Y=\eta} Y^2 \quad (9.47)$$

Thus condition (9.41') is equivalent to $\left(\frac{\partial}{\partial Y} V(0,\underline{s};Y)\right)_{Y=0} = 0$; moreover, the condition in (9.46) is equivalent to $\left(\frac{\partial^2}{\partial Y^2} V(0,\underline{s};Y)\right)_{Y=\eta} Y^2 > 0$ provided that $|Y|$ is sufficiently small. Therefore, (9.47) implies the sufficiency of given conditions in order that (9.45) holds.

Owing to the budget constraint and (9.41'), inequality (9.46) is equivalent to inequality (9.42") between the central second moments.

The operative meaning of Theorem D consists of portfolio selection of assets $\underline{a}$ to cover liabilities $\underline{b}$, immunized with respect to rate risk related to the chance of additive shift. For the stated reasons regarding Theorem B, it is not restrictive, for the sake of simplicity to limit ourselves to the case of two assets and two liabilities. Let the assets be ZCB having unit value U_1 at maturity t_1 and U_2 at maturity $t_2 > t_1$; the liabilities are b_1 at maturity T_1 and b_2 at maturity $T_2 > T_1$. We have to calculate the shares, i.e. the numbers α_1 and α_2 of the asset bond in order to satisfy the budget constraint and the 1st order condition on the durations that are necessary for immunization. Let us agree the unit price $v(0,u) = e^{-\int_0^u \delta(0,z)dz}$ depending on intensity $\delta(0,u)$ and then calculate the value $V(0,\underline{b};0) = \sum_{k=1}^{2} b_k v(0,T_k)$, depending on rates at 0, and the duration $D(\underline{b}) = \sum_{k=1}^{2} T_k b_k v(0,T_k) / \sum_{k=1}^{2} b_k v(0,T_k)$ of liabilities. Then the asset bonds shares are obtained resolving the linear system

$$\begin{cases} \alpha_1 U_1 v(0,t_1) + \alpha_2 U_2 v(0,t_2) = V(0,\underline{b};0) \\ t_1 \alpha_1 U_1 v(0,t_1) + t_2 \alpha_2 U_2 v(0,t_2) = D(\underline{b})\ V(0,\underline{b};0) \end{cases} \tag{9.48}$$

which generalizes system (9.37), as well as its solution

$$\alpha_1 = \frac{V(0,\underline{b};0)(t_2 - D(\underline{b}))}{U_1 v(0,t_1)(t_2 - t_1)}\ ,\ \alpha_2 = \frac{V(0,\underline{b};0)(D(\underline{b}) - t_1)}{U_2 v(0,t_2)(t_2 - t_1)} \quad [25] \tag{9.49}$$

generalizes solution (9.38). In particular, $\underline{b}$ duration takes the place of maturity T of the only b in system (9.37).

In the case of $N = N_1 + N_2$ asset bonds, it is sufficient to consider two subgroups N_1 , N_2 substituting their durations for t_1 and t_2.

Exercise 9.4

Let us consider a portfolio, having liabilities of 50,000 at time 5 and 40,000 at time 7, to cover by shares of two packages of ZCB, the former with U_1=1,000 at maturity 3, the latter with U_2= 800 at maturity 9. We assume that in the market the intensity is $\delta(0,u)$=0,06-0,001u. Let us carry out the immunization and check that it is obtained, applying the Theorem D rules with a check of condition (9.46) on 2[nd] order durations.

A. According to cash-flow distribution and given intensity, we obtain:

- discount factor from u to 0: $v(0,u) = e^{-\int_0^u (0.06-0.001z)dz} = e^{-(0.06u-0.001u^2/2)}$;
- liability value: $V(0,\underline{b};0) = 50000\, e^{-0.2875} + 40000\, e^{-0.3955} = 64440.56$;
- liability duration: $D(\underline{b}) = 5 \cdot 50000 \cdot e^{-0.2875} + 7 \cdot 40000 \cdot e^{-0.3955} = 5.8359$.

The unknowns of the resulting system (9.48) are the real numbers α_1 and α_2 of ZCB shares, which make up the assets. Since

$$t_1 = 3 \quad ; \quad U_1 = 1000\ ; \quad v(0,t_1) = e^{-(0.06\cdot 3 - 0.001\cdot 4.5)} = 0.839037$$
$$t_2 = 9 \quad ; \quad U_2 = 800\ ; \quad v(0,t_2) = e^{-(0.06\cdot 9 - 0.001\cdot 40.5)} = 0.606834$$

the matrix of the coefficients and the constant terms of system (9.48) is given by

$$\begin{Vmatrix} 839.037 & 485.467 & 64,440.56 \\ 25,17.111 & 4,369.203 & 376,068.66 \end{Vmatrix}$$

25 We can observe that: $(\alpha_1>0)\cap(\alpha_2>0) \Leftrightarrow (t_1<D(\underline{b})<t_2)$.

Therefore, owing to (9.49), the shares are

$$\alpha_1 = \frac{64,440.56\ (9-5.8359)}{839.037\ (9-3)} = 40.50206\ ;\alpha_2 = \frac{64,440.56\ (5.8359-3)}{485.467\ (9-3)} = 62.73995$$

The total par value to obtain for the two assets is:

par value (1) = 40,502.06 ; par value (2) = 50,191.96

With such amounts the budget constraint is verified, because

$$V(0,\underline{a};0) = \alpha_1 U_1 v(0,t_1) + \alpha_2 U_2 v(0,t_2) = 40.50206 \cdot 839.037 + 62.73995 \cdot 485.467 =$$
$$= 64440.55 = V(0,\underline{b};0)$$

The equality between durations is also verified. Thus,

$$D(\underline{a}) = (40.50206 \cdot 2517.111 + 62.73995 \cdot 4369.203)/64440.56 = 5.8359 = D(\underline{b})$$

We must now evaluate the 2nd order durations to verify if the immunization sufficient condition is satisfied. We obtain:

$$D^{(2)}(\underline{a}) = (3^2 \cdot 40.50206 \cdot 839.037 + 9^2 \cdot 62.73995 \cdot 485.467)/64440.56 = 43.031249$$
$$D^{(2)}(\underline{b}) = (5^2 \cdot 50000 \cdot e^{-0.2875} + 7^2 \cdot 40000 \cdot e^{-0.3955})/64440.56 = 35.031098$$

Regarding the central second moments, i.e. the variances, of ($\underline{t}'$&$\underline{a}$) and ($\underline{t}''$&$\underline{b}$) we obtain:

$$\sigma^2(\underline{a}) = D^{(2)}(\underline{a}) - \left(D(\underline{a})\right)^2 = 8.973520 \quad ; \quad \sigma^2(\underline{b}) = D^{(2)}(\underline{b}) - \left(D(\underline{b})\right)^2 = 0.973369$$

Therefore, the immunization condition is satisfied. We can verify that the value of $\underline{s}$ is 0 with a relative minimum if the intensity is the given $\delta(0,u) = 0.06-0.001u$, valuing under shift $|Y| = 0.005$. For the sake of simplicity, we assume that the shift occurs in 0^+ only after valuation but this hypothesis is not restrictive: the conclusions also hold with any shift before 3. Valuing after shift, we obtain:

$$\delta(0^+,z)=0.06-0.001z \quad ; \quad v(0^+,u) = e^{-\int_0^u (0.06+Y-0.001z)dz}$$

The statements are: $\omega_1 = (Y = +0.005)$; $\omega_2 = (Y = -0.005)$.

– if $\omega = \omega_1$: $v(0^+,u) = e^{-\int_0^u (0.065-0.001z)dz} = e^{-(0.065u-0.001u^2/2)}$

– if $\omega = \omega_2$: $v(0^+,u) = e^{-\int_0^u (0.055-0.001z)dz} = e^{-(0.055u-0.001u^2/2)}$

The values at 0 under shift in 0^+ are:

– if $\omega = \omega_1$:

$V(0,\underline{a};+0.005) =$

$= 40.50206 \cdot 1000 \cdot e^{-(0.065 \cdot 3 - 0.001 \cdot 4.5)} + 62.73995 \cdot 800 \cdot e^{-(0.065 \cdot 9 - 0.001 \cdot 40.5)} =$

$= 40502.06 \cdot e^{-0.1905} + 50191.96 \cdot e^{-0.5445} = 62594.76;$

$V(0,\underline{b};+0.005) = 50000 \cdot e^{-(0.065 \cdot 5 - 0.001 \cdot 12.5)} + 40000 \cdot e^{-(0.065 \cdot 7 - 0.001 \cdot 24.5)} =$

$= 50000 \cdot e^{-0.3125} + 40000 \cdot e^{-0.4305} = 62588.14;$

$V(0,\underline{s};+0.005) = 62594.76 - 62588.14 = +6.62;$

– if $\omega = \omega_2$:

$V(0,\underline{a};-0.005) =$

$= 40502.06 \cdot e^{-0.1605} + 50191.96 \cdot e^{-0.4545} = 66356.44;$

$V(0,\underline{b};-0.005) = 50000 \cdot e^{-(0.055 \cdot 5 - 0.001 \cdot 12.5)} + 40000 \cdot e^{-(0.055 \cdot 7 - 0.001 \cdot 24.5)} =$

$= 50000 \cdot e^{-0.2625} + 40000 \cdot e^{-0.3605} = 66349.42;$

$V(0,\underline{s};-0.005) = 66356.44 - 66349.42 = +7.02.$

Thus the immunization is checked. Let us verify the different changes of asset and liability values depending on a shift, implying immunization:

– if $\omega = \omega_1$ (δ increases):

(assets) $V(0,\underline{a};+0.005) - V(0,\underline{a};0) = 62594.76 - 64440.56 = -1845.80$

(liabilities) $V(0,\underline{b};+0.005) - V(0,\underline{b};0) = 62588.14 - 64440.56 = -1852.42$

The decrease of the value of the assets is less than the decrease of the value of the liabilities:

– if $\omega = \omega_2$ (δ decreases):

(assets) $V(0,\underline{a};-0.005) - V(0,\underline{a};0) = 66356.44 - 64440.56 = +1915.88$

(liabilities) $V(0,\underline{b};-0.005) - V(0,\underline{b};0) = 66349.42 - 64440.56 = +1908.86$

The increase in the value of the assets is greater than the increase in the value of the liabilities.

We gave the conditions for semi-deterministic immunization of rate risk in several hypotheses, but always with reference to one additive random shift. In the case of several additive shifts, we can carry out subsequent immunizations.

Shiu (1990) generalized the Redington scheme, not only referring to a non-flat rate structure but to *non-additive shifts* $Y(u)$ *with* $u>0$ as well. With regards to this extension, we can prove that to obtain immunization the conditions in (9.45) and (9.46) are needed jointly with other inequality constraints.

However, we do not dwell here on these generalizations and stochastic extensions of the immunization, leaving such questions to be discussed in specialized papers.